Chapter 11

A WALSH-TYPE MULTIRESOLUTION ANALYSIS

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Abstract We introduce a class of orthonormal matrices $U^{(n)}$ of order $p^n \times p^n$, $p = 2, \ldots, n = 1, \ldots$. The construction of those matrices is achieved in different scales by an iteration process, determined by a repetitive block matrix operation, involving the cross product of properly selected sub-matrices. For the case $p = 2$ we get the well known Walsh system. This particular construction also induces a multiscale transform on $L_2(\mathbf{T})$, reminiscent (although different) of a multiresolution analysis of $L_2(\mathbf{T})$.

1. Introduction

In order to provide efficient multiscale analysis on finite data, we seek for linear transforms whose corresponding matrices have the ability to detect specific characteristics from those data. In [4], we introduced a class of weighted sparse matrices for the purpose of prediction of almost periodic time series, while in [5] we built sparse matrices capable of revealing local information at different scales. In [3], we introduced a new class of sparse invertible matrices $H(m)$ of order $m \times m$, suitable for grammar detection of symbolic sequences. In fact, the matrices $H(m)$ may be considered as a generalization of the usual Haar matrices, since their construction was based on dilation and translation operations on unbalanced Haar functions. Thus, we obtained a generalized Haar transform:

$$\{t_n : n = 1, \ldots, m\} \leftrightarrow \{< t, h_n >: n = 1, \ldots, m\},$$

where $<, >$ is the usual inner product of the Euclidean space $\mathbf{R}^m$ and where h_n are the rows of $H(m)$.

In this work we dealt with the problem: what happens if we use dilation and replication operations, instead of using dilation and translation operations on matrices?

In Section 2, we build a discrete transform on finite data by using an iteration in scales. The cross product of matrices plays a central role in our construct, because it can be used either as a dilation or replication operator. So, we start from an initial matrix U of order $p \times p$. In every step of the iteration process we create a new matrix $U^{(n)}$ of order $p^n \times p^n$. $U^{(n)}$ is a block matrix, whose block sub-matrices are defined from the cross product $U^{(n-1)} \otimes U_i$ (see below). In Theorem 11.1, we prove that the matrices $U^{(n)}$ are orthonormal, whenever the initial matrix U is orthonormal. Thus, we obtain a discrete transform:

$$\{t_i : i = 1, \dots, p^n\} \leftrightarrow \{< t, U_i^{(n)} >: i = 1, \dots, p^n\},$$

where $U_i^{(n)}$ are the rows of $U^{(n)}$. For a suitable selection of the matrix U we see that the resulting orthonormal system is the Walsh system.

Since to any row of the matrix $U^{(n)}$ there corresponds a step function on $\mathbf{T}$, an orthonormal set $\widetilde{M}_n = \{\widetilde{m}_k(\gamma) : k = 1, \dots, p^n\}$ of functions of $L_2(\mathbf{T})$ emerges naturally from the matrix $U^{(n)}$. In Section 3 we see that the set $\widetilde{M}_n$ is produced by successive dilations and replicas of a generator set of functions $M = \{m_i(\gamma) : i = 0, \dots, p-1\}$:

$$m_i(\gamma) = \sum_{j=1}^{p} U_{i+1,j} \mathbf{1}_{\left[\frac{j-1}{p}, \frac{j}{p}\right)}, \; i = 0, \dots, p-1.$$

Indeed:

$$\widetilde{M}_n = \left\{ \widetilde{m}_k(\gamma) = \prod_{j=0}^{n-1} m_{\varepsilon_j}(p^j \gamma) : \; k = 1 + \sum_{j=0}^{n-1} \varepsilon_j p^j, \; \varepsilon_j \in \{0, \dots, p-1\} \right\}.$$

Finally, we see that our multiscale construction naturally extends to an invertible transform on $L_2(\mathbf{T})$.

2. A class of Walsh-type discrete transforms

Notation: Let $M_{n,m}$ be the set of all matrices of order $n \times m$ over the field of complex numbers. If $n = m$, then $M_{n,m}$ is abbreviated to M_n. We shall use the symbolism $A = [A_{ij}]$ to denote a matrix A with elements A_{ij}. The notation

$$A_i = \{A_{i,j} : j = 1, \dots, m\}$$

shall be used to denote the i-row of a matrix A. We define the following operators:

DEFINITION 11.1 *For $p = 2, \ldots,$ the tensor product of two matrices $A \in M_{n,m}$ and $B \in M_{k,l}$ is a block matrix $A \otimes B \in M_{nk,ml}$:*

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix}.$$

DEFINITION 11.2 *Let $S : M_{n_1,m} \times \ldots \times M_{n_k,m} \to M_{n_1+\ldots+n_k,m}$ be the following block matrix operator:*

$$S(M_1, \ldots, M_k) = \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix}.$$

DEFINITION 11.3 *Let U be an orthonormal matrix of order $p \times p$, we define a sequence of block matrices $U^{(n)}$, where $n = 1, ..., N$, by using the following iteration:*

$$U^{(n)} = \begin{cases} U, & n = 1 \\ S\left(U^{(n-1)} \otimes U_1, \ldots, U^{(n-1)} \otimes U_p\right), & n = 2, \ldots, N \end{cases}, \qquad (11.1)$$

where U_i is the i row of U.

THEOREM 11.1 *The matrix $U^{(n)}, (n = 1, \ldots, N)$ is orthonormal.*

Proof. We work inductively. Clearly, the theorem is true for $n = 1$. We suppose that the matrix $U^{(n-1)}$ is orthonormal, so it suffices to prove that $< U_j^{(n)}, U_l^{(n)} >= \delta_{j,l}$, where $<,>$ is the usual inner product of the Euclidean space $\mathbf{R}^{p^n}$. Let $j = mp^{n-1} + \zeta,\ l = qp^{n-1} + \sigma$, where $m, q = 0, \ldots, p - 1,\ \zeta, \sigma = 1, \ldots, p^{n-1}$, then:

$$\begin{aligned} \langle U_j^{(n)}, U_l^{(n)} \rangle &= \sum_{r=1}^{p^n} U_{jr}^{(n)} U_{rl}^{(n)} = \sum_{\nu=0}^{p-1} \sum_{\mu=1}^{p^{n-1}} U_{j,\nu p^{n-1}+\mu}^{(n)} U_{\nu p^{n-1}+\mu,l}^{(n)} \\ &= \sum_{\nu=0}^{p-1} \sum_{\mu=1}^{p^{n-1}} U_{m+1,\nu+1} U_{\zeta,\mu}^{(n-1)} U_{\nu+1,q+1} U_{\mu,\sigma}^{(n-1)} \\ &= \left(\sum_{\nu=1}^{p} U_{m+1,\nu} U_{\nu,q+1} \right) \left(\sum_{\mu=1}^{p^{n-1}} U_{\zeta,\mu}^{(n-1)} U_{\mu,\sigma}^{(n-1)} \right) \\ &= \delta_{m,q} \delta_{\zeta,\sigma} = \delta_{j,l}. \end{aligned}$$

It is clear that the inverse matrix of $U^{(n)}$ coincides to its transpose $\left(U^{(n)}\right)^T$. The following multiresolution structure arises from the matrices $U^{(n)}$:

Let V_{p^n} be the space of all real-valued sequences of length p^n and let $U_i^{(n)}$ be the i-row of the matrix $U^{(n)}$, then any element $t \in V_{p^n}$ can be written as:

$$t_l = \sum_{i=1}^{p^n} \langle t, U_i^{(n)} \rangle U_{i,l}^{(n)}.$$

Let $j = 1, \ldots, n-1,\ k = 1, \ldots, p-1$, we define

$$W_{j,k} = \text{span}\{U_{kp^j+s}^{(n)} : s = 1, \ldots, p^j\},$$

then, we have the decomposition:

$$V_{p^n} = V_0 \oplus_{j=1}^{n-1} \oplus_{k=1}^{p-1} W_{j,k},$$

where $V_0 = \text{span}\{U_s^{(n)} : s = 1, \ldots, p\}$.

EXAMPLE 11.1 *Let $p = 3,\ n = 3$, then $V_{3^3} = V_0 \oplus_{j=1}^{2} \oplus_{k=1}^{2} W_{j,k}$, where:*

$$V_0 = \textit{span}\{U_1^{(3)}, \ldots, U_3^{(3)}\},$$

$$W_{1,1} = \textit{span}\{U_4^{(3)}, \ldots, U_6^{(3)}\}, \quad W_{1,2} = \textit{span}\{U_7^{(3)}, \ldots, U_9^{(3)}\},$$
$$W_{2,1} = \textit{span}\{U_{10}^{(3)}, \ldots, U_{18}^{(3)}\}, \quad W_{2,2} = \textit{span}\{U_{19}^{(3)}, \ldots, U_{27}^{(3)}\}.$$

DEFINITION 11.4 *Let $p \geq 2$, we define the following matrix $\Psi^{(p)}$ of order $p \times p$:*

$$\psi_{ij}^p = \begin{cases} \frac{1}{\sqrt{p}}, & \textit{whenever } i = 1 \\ \frac{1}{\sqrt{p-i+1}} \frac{1}{\sqrt{p-i+2}}, & \textit{whenever } 1 \leq j \leq p-i+1 \\ -\frac{\sqrt{p-i+1}}{\sqrt{p-i+2}}, & \textit{whenever } j = p-i+2, \\ 0, & \textit{whenever } p-i+2 < j \leq p \end{cases} \quad i, j = 1, \ldots, p.$$

EXAMPLE 11.2 $\Psi^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \Psi^3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\sqrt{2} \\ \frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} & 0 \end{pmatrix}.$

PROPOSITION 11.1 *(see [2])*
The matrix $\Psi^{(p)}$ satisfies the following properties:

(i) $\sum_{j=1}^{p} \psi_{ij}^{(p)} = 0,\ i = 2, \ldots, p.$

(ii) $\psi_i^{(p)}\psi_j^{(p)} = \psi_{i,1}^{(p)}\psi_j^{(p)}$, *whenever* $i < j, i, j = 1, \ldots, p$.

(iii) *The matrix* $\Psi^{(p)}$ *is orthonormal.*

OBSERVATION 11.1 *If we consider the iteration (11.1) with initial matrix* $U = \Psi^{(2)}$, *then we obtain the Walsh system (see [6]). Indeed:*

$$U^{(1)} = \Psi^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, U^{(2)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ldots.$$

Whenever $p > 2$, *we get a Walsh-type construction.*

OBSERVATION 11.2 *If we consider the iteration (11.1) with initial matrix* $U = \Psi^{(p)} = \left(e^{2\pi ikl/p}\right)_{k,l=0}^{p-1}$, *then we obtain the Generalized Walsh system as defined in [6].*

3. A multiscale transform on $L_2(\mathrm{T})$

We denote by $\mathbf{R}$ the additive group of real numbers and by $\mathbf{Z}$ the subgroup consisting of the integers. The group $\mathbf{T}$ is defined as the quotient $\mathbf{R}/\mathbf{Z}$. Since there is an obvious identification between functions on $\mathbf{T}$ and 1-periodic functions on $\mathbf{R}$, from now on we identify the elements of the space $L_2(\mathbf{T})$ of all complex valued Lebesgue square integrable functions on $\mathbf{T}$, as 1-periodic functions on $\mathbf{R}$.

Since any row $U_k^{(n)}$, $k = 1, \ldots, p^n$ of the matrix $U^{(n)}$ defined in Theorem 11.1 can be assigned to a step function $\widetilde{m}_k(\gamma)$ on $\mathbf{T}$ such that

$$\widetilde{m}_k(\gamma) = m_{kj},\ \gamma \in \Omega_{j,n} = \left[\frac{j-1}{p^n}, \frac{j}{p^n}\right),\ j = 1, \ldots, p^n,$$

an orthonormal set of functions of $L_2(\mathbf{T})$ emerges naturally from the construction presented in section 2:

$$\widetilde{M}_n = \left\{\widetilde{m}_k(\gamma) : \widetilde{m}_k(\gamma) = \sum_{j=1}^{p^n} U_{k,j}^{(n)} \mathbf{1}_{\Omega_{j,n}}(\gamma),\ k = 1, \ldots, p^n.\right\}$$

Moreover, if U is the initial orthonormal matrix of the iteration process (11.1), by defining:

$$m_i(\gamma) = \sum_{j=1}^{p} U_{i+1,j}\mathbf{1}_{\Omega_{j,n}}(\gamma),\ i = 0, \ldots, p-1,$$

we can see that the set $\widetilde{M}_n$ can be produced by successive dilations of the functions $m_i(\gamma)$ in the following:

$$\widetilde{M}_n = \left\{ \widetilde{m}_k(\gamma) = \prod_{j=0}^{n-1} m_{\varepsilon_j}(p^j\gamma),\ k = 1 + \sum_{j=0}^{n-1} \varepsilon_j p^j,\ \varepsilon_j \in \{0, \dots, p-1\} \right\}. \tag{11.2}$$

Moreover, we can prove:

THEOREM 11.2 *Let* $\{V_n : V_n \subset V_{n+1},\ n \geq 1\}$ *be a nested sequence of* p^n*-dimensional subspaces of* $L_2(\mathbf{T})$*, whose orthonormal basis is the set* $\widetilde{M}_n$ *defined in (11.2), then:*

$$\overline{\cup_{n\geq 1} V_n} = L_2(\mathbf{T}).$$

Proof. See [1].

Acknowledgments

Research supported by the Joint Research Project within the Bilateral S&T Cooperation between the Hellenic Republic and the Republic of Bulgaria (2004-2006).

References

[1] N. Atreas and Antonis Bisbas, "On a class of Generalized Riesz Products derived from a multiscale transform", preprint.

[2] N. Atreas and C. Karanikas, "Haar-type orthonormal systems, data presentation as Riesz Products and a recognition on symbolic sequences", *Proceedings of the Special Session on Frames and Operator Theory in Analysis and Signal Processing*, to appear in *Contemp. Math.*, Vol. 451, (2008).

[3] N. Atreas and C. Karanikas, "Multiscale Haar unitary matrices with the corresponding Riesz Products and a characterization of Cantor - type languages", *Fourier Anal. Appl.*, 13, 2, (2007), 197-210.

[4] N. Atreas, C. Karanikas and P. Polychronidou, "A class of sparse Unimodular Matrices generating Multiresolution ans Sampling analysis for data of any length", accepted to *SIAM J. Matrix Anal. Appl.*

[5] N. Atreas and P. Polychronidou, "A class of sparse invertible matrices and their use for non-linear prediction of nearly periodic time series with fixed period", to appear in *Numer. Funct. Anal. Optim.*, Vol. 29, Issues 1&2, (2008).

[6] Golubov B., Efimov A. and Skvortsov V., *Walsh Series and Transforms. Theory and Applications*, Kluwer Academic Publishers, New York, (1991).