Chapter 8

APPLICATIONS OF SIDON TYPE INEQUALITIES

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Abstract The aim of this paper is to show several areas, such as integrability conditions, strong summation, theory of multipliers, where the so called Sidon type inequalities can be applied. We will focus our attention mainly to the methods that link these areas and the Sidon type inequalities, and present some characteristic results. The bibliography at the end of the paper is far from complete but provides enough information for those who would like to learn more about these fields. The model used for the interpretation is the Walsh case but we will make remarks concerning the trigonometric case as well.

1. Introduction

Throughout this paper w_n will denote the nth Walsh function in Paley's ordering. $\widehat{f}(n)$, $S_n f$, and Sf $(f \in L^1)$ will stand for the nth Walsh-Fourier coefficient, the nth Walsh-Fourier partial sum, and Walsh-Fourier series respectively. The Walsh-Dirichlet kernels are defined as $D_n = \sum_{k=0}^{n-1} w_k$. Concerning the theory of dyadic analysis the book of Schipp, Wade and Simon [SWS90] is recommended as a general refence.

Hardy spaces will play an important role in the paper. Especially the real non periodic Hardy space $\mathcal{H}$, and the dyadic Hardy space $\mathbb{H}$. Basic results results concerning Hardy spaces can be found in the book of Kashin and Saakian [KS89]. For details on the theory of dyadic Hardy spaces and their applications we refer the reader to the monography of Weisz [Wei94]

Another type of Banach spaces that will be used in the paper are Orlicz spaces. If M is a so called Young function then L_M will denote the corresponding Or-

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licz space. The monographies of Krasnosel'skiĭ and Rutickiĭ [KR61], and of Rao and Zen [RZ91] are recommended for the theory and application of Orlicz spaces.

2. Sidon Type Inequalities

Let $c_k \in \mathbb{R}$ $(k = 1, \ldots, 2^n,\ n \in \mathbb{N})$. Then the step function $\Gamma(c_1, \ldots, c_{2^n})$ is defined as follows

$$\Gamma(c_1, \ldots, c_{2^n})(x) = c_k \qquad \Big(\frac{k-1}{2^n} \le x < \frac{k}{2^n},\ k = 1, \ldots, 2^n\Big).$$

Suppose that X is a norm in the space of dyadic step functions on $[0, 1)$.
An inequality

$$\frac{1}{2^n}\Big\| \sum_{k=1}^{2^n} c_k D_k \Big\|_1 \le C_X \|\Gamma(c_1, \ldots, c_{2^n})\|_X \qquad (c_k \in \mathbb{R},\ n \in \mathbb{N}) \tag{8.1}$$

is called a Sidon type inequality.
Such an inequality was first proved by Telyakovskiĭ [Tel73] for trigonometric Dirichlet kernels with $X = L^\infty$, i.e. with $\|\Gamma(c_1, \ldots, c_{2^n})\|_X = \max_{1\le k\le 2^n} |c_k|$. His result was improved by Bojanić and Stanojević [BS82] by showing that (8.1) holds for $X = L^p$ $(1 < p \le \infty)$. Another improvement, where $X = \log \alpha L_1 + \alpha^{-1/q} L_p$ $(\alpha \ge 1,\ \frac{1}{p} + \frac{1}{q} = 1)$, is due to Tanović-Miller [Tan90]. The best result along this line was given by Schipp [Sch92]. Namely, he showed that the real non periodic Hardy space $\mathcal{H}$ can stand for X in (8.1).
Móricz and Schipp [MS90] proved the Walsh analogue of the result of Bojanić and Stanojević. Then Schipp [Sch92] proved that (8.1) holds for the Walsh-Dirichlet kernels if X is the dyadic Hardy space $\mathbb{H}$. We note that Schipp considered Sidon type inequalities and Hardy norms in a more general setting, and the Walsh and the trigonometric systems are special cases only. Later the author showed [Fri00] that even in the Walsh case X can be the real non periodic Hardy space $\mathcal{H}$.
At this point we call the attention that unlike the other norms applied in Sidon type inequalities the Hardy norm is no rearrangement invariant, and there is no simple direct expression of c_k's that provides the Hardy norm of the step function $\Gamma(c_1, \ldots, c_{2^n})$. Another motivation for further investigations was to find out how far is the Hardy norm variant from the best possible result. In connection with these the author [Fri95] proved that

$$\begin{aligned}\max_{p\in P_{2^n}} \Big\| \sum_{k=1}^{2^n} c_{p_k} D_k \Big\|_{L_1} &\approx \|\Gamma(c_1, \ldots, c_{2^n})\|_{L_M} \\ &\approx \sum_{k=1}^{2^n} |c_k| \Big(1 + \log^+ \frac{|c_k|}{2^{-n}\sum_{j=1}^{2^n} |c_j|}\Big),\end{aligned} \tag{8.2}$$

where P_{2^n} is the set of permutations of $\{1, \ldots, 2^n\}$, and L_M is the Orlicz space induced by the Young function $M(x) \approx x \log x$.
We note that similar result holds for the trigonometric system ([Fri93]).
This means that (8.2), in which the right side is an Orlicz norm, is the best rearrangement invariant Sidon type inequality. Also the connection with the Hardy norms and this Orlicz norm was given in [Fri93]. Namely, it was shown that the largest rearrangement invariant space included in $\mathcal{H}$ is the Orlicz space L_M. More precisely, if

$$H^\sharp = \big\{ f \in \mathcal{H} \ : \ f \circ \nu \in \mathcal{H}, \ \nu \in L_0 \big\},$$

where L_0 is the set of one-to-one measure preserving maps on $[0,1)$ then

$$H^\sharp = L_M \,.$$

Moreover

$$\|f\|_{H^\sharp} = \sup\big\{ \|f \circ \nu\|_{\mathcal{H}} \ : \ \nu \in L_0 \big\} \approx \|f\|_{L_M} \approx \int_0^1 |f| \Big(1 + \log^+ \frac{|f|}{\|f\|_1} \Big) \,.$$

There have several generalizations and variants been given of the Sidon type inequalities above. For example the shifted variant of (8.2), i.e. the one that has $\sum_{k=K}^N c_k D_k$ $(K, N \in \mathbb{N})$ on the left side is in [Fri95]. Another version that has found applications is a truncated Sidon type inequality proved by Móricz [Mór90]. Its Walsh version, which was shown by Daly and the author [DF03], reads as follows

$$\int_{2^{-N}}^1 \Big| \sum_{k=1}^{2^n} c_k D_k(x) \Big| \, dx \le C 2^{N(1-1/q)} \Big(\sum_{k=1}^{2^n} |c_k|^q \Big)^{1/q}, \qquad (8.3)$$

where $(n, N \in \mathbb{N}, \ 1 < q \le 2)$.

As a closing remark to this section we note that however the development of Sidon type inequalities went parallel with respect to the trigonometric and the Walsh systems the techniques used for the two cases are different at several points.

3. Integrability Classes

We will consider the integrability and L^1-convergence of the Walsh series

$$\sum_{n=0}^\infty a_n w_n \,, \qquad (8.4)$$

where (a_n) is a sequence of real numbers.
Let $\widehat{L}_W$ denote the space of real sequences for which (8.4) represents a Walsh-Fourier series of an integrable function. There is no characterization known for

$\widehat{L}_W$ in terms of the coefficients in (8.4). There are several known examples for conditions with respect to the coefficients that imply the integrability of (8.4). A subset of $\widehat{L}_W$ generated by such a condition is called integrability class.
The existence of an integrable function whose Walsh-Fourier series is (8.4) does not mean that the series converges to the function in L^1 norm. An integrability class is called an integrability and convergence class if for each element (a_n) of it the corresponding series converges in L^1-norm if and only if $\lim_{n\to\infty} |a_n| \, \|D_n\|_1 = 0$.

The connection between Sidon type inequalities and integrability and L^1-convergence classes is quite obvious. Indeed, a simple summation by parts of (8.4) leads to the series

$$\sum_{k=1}^{\infty} \Delta a_k D_k \, .$$

After breaking it into dyadic blocks a proper condition for $\| \sum_{k=2^n}^{2^{n+1}-1} \Delta a_k D_k \|_1$ would yield the desired convergence. For instance the well known integrability and convergence class due to Fomin [Fom73] in the trigonometric case, i.e. the one given by the condition

$$\sum_{n=0}^{\infty} 2^{n(1-1/p)} \Big(\sum_{k=2^n}^{2^{n+1}-1} |\Delta a_k|^p \Big)^{1/p} \le C \quad (p > 1) \, .$$

can be deduced from the Sidon type inequalities of Bojanić and Stanojević [BS82] in the trigonometric and of Schipp and Móricz [MS90] in the Walsh case.
Let us mention some other classical integrability and convergence conditions. Young [You13] showed that the set of convex null sequences forms an integrability and convergence class. The same holds for the set of so called quasi convex sequences as was shown by Komogorov [Kol23]. Another condition was given by Sidon [Sid39]. His result was reformulated by Telyakovskiĭ [Tel73]. This class contains those null sequences (a_k) for which there exists a non increasing sequence (A_k) such that $\sum_{k=0}^{\infty} A_k < \infty$, and $|\Delta a_k| \le A_k$. It was shown by he author [Fri96] that also these classical results can be deduced from proper known Sidon type inequalities.
We note that there exist integrability and convergence conditions that can not be originated from a Sidon type inequality. In connection with such results we refer the reader to the papers of Aubertin and Fournier [AF93], [AF94] and of Buntinas and Tanović-Miller [BT90]. There is yet another well-known condi-

tion which turned out to be connected with a Sidon type inequality. Namely,

$$\sum_{n=2}^{\infty}\Big|\sum_{k=1}^{[n/2]}\frac{\Delta a_{n-k}-\Delta a_{n+k}}{k}\Big|<\infty \quad , \quad \sum_{n=1}^{\infty}|\Delta a_n|<\infty\,. \tag{8.5}$$

is an integrability and convergence condition due to Telyakovskiĭ [Tel64]. We remark that it is an improvement of an earlier condition of Boas [Boa56], and was proved for cosine series originally.

The author [Fri01] proved that the formula in (8.5) can be understood as a sequence norm. More precisely, it is equivalent to a sequence Hardy norm which has an atomic structure, and the atoms there are related to those of $\mathcal{H}$. This interpretation of (8.5) made possible the extension of the Telyakovskiĭ condition to other cases including Walsh series.

Let us now suppose that $\sum_{n=0}^{\infty} a_n w_n$ is a Walsh-Fourier series, i.e. $a_k = \widehat{f}_k$ with some $f \in L^1$, and consider its L^1 convergence. Comparing it with the previous setting here the integrability is already guaranteed and only the L^1 convergence is the question. A major difference is that in this case the Fejér means of the Walsh-Fourier series are convergent. So are the generalized de la Vallée Poussin means

$$V_{n,\lambda}f=\frac{1}{[\lambda n]-n+1}\sum_{k=n}^{[\lambda n]}S_kf \qquad (n\in\mathbb{P},\ \lambda>1,\ f\in L_1). \tag{8.6}$$

Then it is enough to provide condition for the convergence of $V_{n,\lambda}f - S_nf$. By manipulating this difference the left side of the Sidon inqualities (8.1) will come up. This is how the L^1 convergence problem is linked to Sidon type inequalities. Typical results in this line are the so called Hardy-Karamata type Tauberian conditions. For instance Bojanić and Stanojević [BS82] showed that if

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}k^{p-1}|\Delta\widehat{f}(k)|^p=0 \qquad (p>1)$$

then $\lim_{n\to\infty}\|f-S_nf\|_1=0$ if and only if $\lim_{n\to\infty}\widehat{f}_n\log n=0$. The Walsh version was proved by Móricz [Mór89].

Another condition, namely

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}|\Delta\widehat{f}(k)|\log k=0\,.$$

was given by Grow and Stanojević [GS95].

Then the author [Fri97] gave the following condition, which subsumes the previous ones

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}|\Delta\widehat{f}(k)|\log^+|k\Delta\widehat{f}(k)|=0\,. \tag{8.7}$$

If (8.7) holds then $\lim_{n\to\infty}\|f-S_nf\|_{L_1}=0$ if and only if $\lim_{n\to\infty}\widehat{f}(n)L_n=0$, where L_n stands for the nth Walsh-Lebesgue constant. Moreover if $f\in\mathbb{H}$ then $\lim_{n\to\infty}\|f-S_nf\|_{\mathbb{H}}=0$ if and only if $\lim_{n\to\infty}\widehat{f}(n)L_n=0$. We note that the trigonometric version of (8.7) can be found in [Fri97'].

4. Strong Summation

Let us start this section with the classical result of Hardy and Littlewood [HL13] on strong summability of trigonometric Fourier series of continuous functions:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|S_kf(x)-f(x)|^q=0 \qquad (q>1). \tag{8.8}$$

The connection between strong summability of continuous functions and Sidon type inequalities relies on a duality relation. Suppose that X is a homogeneous Banach spaces that contains the set of dyadic step functions. Let Y denote the dual of X.

The Y strong means of a continuous function f at x is defined as

$$\big\|\Gamma\big(S_1f(x),\dots,S_{2^n}f(x)\big)\big\|_Y.$$

For instance the strong mean in (8.8) corresponds to $Y=L^q$.

The following duality result is proved in [FS95], and [FS98] in a more general setting:

$$\begin{gathered}\sup_x\big\|\Gamma\big(S_1f(x),\dots,S_{2^n}f(x)\big)\big\|_Y\le C_X\|f\|_{L_\infty}\\ \text{if and only if}\\ \Big\|\frac{1}{2^n}\sum_{k=1}^{2^n}c_kD_k\Big\|_{L_1}\le C_X\|\Gamma(c_1,\dots,c_{2^n})\|_X\,.\end{gathered} \tag{8.9}$$

Here f is a continuous function in the trigonometric case, while dyadically continuous function in the Walsh case, and the c_k-s are arbitrary real numbers. For example the trigonometric Sidon type inequality of Bojanic and Stanojević [BS82] is dual to the strong summability result of Hardy and Littlewood [HL13]. Concerning strong summation and approximation by trigonometric Fourier series we refer the reader to the monograph of Leindler [Lei85] as a general reference.

If L_M is the Orlicz space induced by the Young function $M(x)\approx x\log x$ then its dual is L_N where $N = e^x - 1$. On this bases we deduced [FS98] from the Sidon inequality in (8.2) that for any $A > 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(e^{A|S_k f(x) - f(x)|} - 1\right) = 0 \tag{8.10}$$

in both the trigonometric and the Walsh cases. Moreover, utilizing the fact that (8.2) is the best rearrangement invariant Sidon type inequality for the trigonometric and the Walsh systems we could prove [FS98] the sharpness of (8.10) in the following sense: If $\varphi \nearrow$, $\varphi(0) = 0$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(|S_k f(x) - f(x)|) = 0 \tag{8.11}$$

then there exists $A > 0$ such that $\varphi(x) \le e^{Ax}$. We note that (8.10) and its sharpness was originally proved by Totik [Tot80] for continuous φ in (8.11).
The rest of this section is on strong approximation properties of Walsh, and trigonometric Fourier series. First we note that the investigation of strong approximation properties of trigonometric Fourier series was started by Alexits and Králik [AK63]. Let us start with modifying the notation of de la Vallée Poussin means given in (8.6) as follows

$$V_{r,n} f = \frac{1}{r} \sum_{k=1}^{r} S_{k+n} f \qquad (r, n \in \mathbb{N}).$$

Let Y be as before. Then we can define the concept of strong Y oscillations of Fourier partial sums as follows

$$\left\|\Gamma\left(S_{1+n} f(x) - V_{1,n} f(x), \ldots, S_{r+n} f(x) - V_{r,n} f(x)\right)\right\|_Y \qquad (r, n \in \mathbb{N}).$$

The error of best approximation of f by Walsh polynomials of order at most n is defined as follows

$$E_n f = \inf_{p \in \mathcal{P}_n} \|f - p\|_\infty \qquad (n \in \mathbb{P}),$$

where $\mathcal{P}_n$ is the set of Walsh polynomials of order not greater than n.
Schipp and the author [FS98] showed that that the following duality relation

holds between strong oscillation and shifted Sidon type inequalities.

$$\frac{1}{2^n}\|\sum_{k=1}^{2^n} c_k D_{k+j2^n}\|_1 \le C_X \|\Gamma(c_1,\dots,c_{2^n})\|_X$$

$$\text{with} \quad \sum_{k=1}^{2^n} c_k = 0 \quad (j,n\in\mathbb{N})$$

if and only if

$$\big\|\Gamma\big(S_{1+j2^n}f(x)-V_{2^n,j2^n}f(x),\dots,S_{(j+1)2^n}f(x)-V_{2^n,j2^n}f(x)\big)\big\|_Y \le C_Y E_{j2^n}f \quad (j,n\in\mathbb{N})\,. \tag{8.12}$$

On the basis of this duality relation all we need is to estimate $|f(x)-V_{2^n,j2^n}f(x)|$ in order that we can deduce estimations for the generalized strong means, i.e. for

$$\big\|\Gamma\big(S_{j2^n+1}f(x)-f(x),\dots,S_{(j+1)2^n}f(x)-f(x)\big)\big\|_Y \qquad (n\in\mathbb{N})\,.$$

In other words the only additional information needed is the rate of convergence of generalized de la Vallée Poussin means.
There are several results concerning strong approximation that can be deduced from (8.12). Here we mention only one and refer the reader for further results to [FS98].
Let $\varphi:[0,\infty)\mapsto\mathbb{R}$ be a monotonically increasing continuous function with $\lim_{u\to 0^+}\varphi(u)=0$ for which there exists A such that

$$\varphi(u)\le\exp(Au) \qquad (u\ge 0)\,,$$

and

$$\varphi(2u)\le A\varphi(u) \qquad (0<u<1).$$

Then

$$\frac{1}{n}\sum_{k=n+1}^{2n}\varphi\big(\big|S_kf(x)-f(x)\big|\big)\le C\varphi(E_nf) \qquad (n\in\mathbb{P}).$$

The trigonometric version of this estimate was given by Totik in [Tot85].

5. Multipliers

Let φ be a sequence of real numbers. Then the multiplier operator T_ϕ is defined as

$$T_\phi f=\sum_{k=0}^{\infty}\phi(k)\widehat{f}(k)w_k \qquad (f\in L^1).$$

The operator T_ϕ can be considered as a convolution operator with generalized kernel $\sum_{k=0}^{\infty} \phi(k) w_k$. It is known that the operator norm from L^1 to itself can be estimated by the L^1 norm of the kernel. This is how integrability conditions, and therefore Sidon type inequalities are connected with L^1 multiplier conditions. For instance the Telyakovskiĭ type integrability condition proved in [Fri01] implies the multiplier condition [DF03]:
If

$$\sum_{n=1}^{\infty} |\Delta\phi(n)| + \sum_{n=2}^{\infty} \Big| \sum_{k=1}^{[n/2]} \frac{\Delta\phi(n-k) - \Delta\phi(k+n)}{k} \Big| < \infty$$

then T_ϕ is bounded on L^1 and on $\mathbb{H}$.
In connection with this we note that the well known Marcinkiewicz condition [Mar39], i.e.

$$\phi \text{ bounded, and } \sum_{k=2^n}^{2^{n+1}-1} |\Delta\phi(k)| < C$$

is sufficient for the boundedness of T_ϕ on L^p $(1 < p < \infty)$ but is not for L^1 in the trigonometric case. Wo-Sang Young [WSY94] showed this result extends to the setting of Vilenkin groups; and thus, in particular valid in the Walsh case. Daly and the author [DF03] showed that even the stronger condition that ϕ is of bounded variation does not imply that T_ϕ is bonded on L^1 or on $\mathbb{H}$.
Let us turn to the problem of boundedness of multiplier operators on $\mathbb{H}^p$ $(0 < p < 1)$. We will consider the classical coefficient condition, the so called Hörmander-Mihlin condition [Hör60] and [Mih56]. It is known to be sufficient in the spaces L^p $(1 < p < \infty)$ even in the multidimensional case. For details we refer the reader to the monograph of Edwards and Gaudry [EG77]. The condition reads as follows

$$2^j \left(\sum_{k=2^j+1}^{2^{j+1}} \frac{|\Delta\phi(k)|^r}{2^j} \right)^{1/r} \leq C \qquad (j \in \mathbb{N}). \tag{8.13}$$

We note that a multiplier that satisfies the Hörmander-Mihlin condition with any $r \geq 1$ satisfies also the Marcinkiewicz condition and that $L^p = \mathbb{H}^p$ for $p > 1$.
Daly and the author [DF03] showed that unlike the Marcinkiewicz condition the the Hörmander-Mihlin condition extends to Hardy spaces $\mathbb{H}^p$ for $p < 1$. Namely we proved that if ϕ is bounded and satisfies (8.13) with $1 < r \leq 2$, and $p > r/(2r-1)$ then T_ϕ is bounded from $\mathbb{H}^p$ to itself. This in particular means that for any $1 < r \leq 2$ (8.13) is sufficient for the boundedness of T_ϕ on $\mathbb{H}^1$. On the negative side we showed that for any $1 \leq r \leq \infty$ there exists a bounded multiplier φ that satisfies (8.13) with the natural modification for $r = \infty$, but T_φ is not bounded on L_1. This result shows a major difference

between L^1 and $\mathbb{H}^1$.
We also proved in [DF03] hat our result is sharp in the sense that if $p < r/(2r-1)$ then there exists a bounded multiplier ϕ that satisfies (8.13) but T_φ is not bounded from $\mathbb{H}^p$ to L_p. For the trigonometric version of these results see the paper of Daly and the author [DF05].
We note that existing multiplier theorems for Hardy spaces give growth conditions on the dyadic blocks of the Walsh series of the kernel, see e.g. Daly and Phillips [DPh98], [DPh98'], Kitada [Kit87], Onneweer and Quek [OQ89], and Simon [Sim85], but these growth are not computable directly in terms of the coefficients.

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