Chapter 1

FOURIER ANALYSIS IN A SPACE OF CHARACTERISTIC FUNCTIONS OF SUBSETS OF THE RATIONAL NUMBERS

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ABSTRACT. The conventional dyadic algebra $\mathbb{F}$ (as in [3]) may be notated as a division algebra over the field of two elements, with the integer singletons as basal units. The space of characteristic functions of the elements of $\mathbb{F}$ admits an analogue of Fourier analysis, extended here to the function space induced similarly by the $\mathbb{F}$-like algebra having the rational singletons as basal units.

1. INTRODUCTION: THE REAL DYADIC ALGEBRA

The function space of the title is isomorphic to the vector space underlying a division algebra $\mathbb{U}$ whose elements are linear combinations

$$A = \sum_{x \in \mathbb{R}} \varphi_A(x) \{x\}$$

of real singletons $\{x\}$ with coefficients

$$\varphi_A(x) \in \Phi := \{\varnothing, \{0\}\}.$$

The *sum* $A + B$ of $A, B \in \mathbb{U}$ is defined as the symmetric difference

$$A + B = A \vartriangle B = (A \cup B) - (A \cap B).$$

Such summation is extensible to a finite set of addends, and even to convergent infinite series of terms in $\mathbb{U}$. In particular, the sum of a set of pairwise disjoint elements of $\mathbb{U}$ is equal to their union. Thus, for example,

$$\sum_{x \in A} \{x\} = \bigcup_{x \in A} \{x\} = A.$$

The *product* of basal units $\{x\}, \{y\}$ is defined by

$$\{x\}\{y\} = \{x + y\}.$$

We also define

$$\{x\} \varnothing = \varnothing\varnothing = \varnothing \{x\} = \varnothing.$$

With the addition and multiplication thus defined, Φ is isomorphic to the field of two elements. Since

$$\sum_{x \in A} \{x\} = A = \sum_{x \in \mathbb{R}} \varphi_A(x) \{x\} = \sum_{\varphi_A(x) = \{0\}} \varphi_A(x) \{x\} = \sum_{\varphi_A(x) = \{0\}} \{x\}$$

and the set $\{\{x\} : x \in \mathbb{R}\}$ is linearly independent, we have

$$x \in A \quad \text{iff} \quad \varphi_A(x) = \{0\}.$$

Thus $\varphi_A(x)$ is the value at x of the *characteristic function* $\varphi_A : \mathbb{R} \to \Phi$ of A.

The function $\varphi : 2^{\mathbb{R}} \to \Phi^{\mathbb{R}}$ that takes each $A \subseteq \mathbb{R}$ to its characteristic function φ_A is a bijection. Its inverse $\varphi^{-1} : \Phi^{\mathbb{R}} \to 2^{\mathbb{R}}$ takes φ_A to

$$A = \{x \in \mathbb{R} : \varphi_A(x) = \{0\}\}.$$

The characteristic function φ_{A+B} satisfies

$$\sum_{x\in\mathbb{R}} \varphi_{A+B}(x)\{x\} = A + B = \sum_{x\in\mathbb{R}} \varphi_A(x)\{x\} + \sum_{x\in\mathbb{R}} \varphi_B(x)\{x\},$$

which implies, by linear independence, that

$$\varphi_{A+B}(x) = \varphi_A(x) + \varphi_B(x).$$

Thus the sum of the subsets A, B of $\mathbb{R}$ is mapped by φ to the pointwise sum $\varphi_A + \varphi_B$ of their characteristic functions.

The function φ_{AB} satisfies

$$\sum_{x\in\mathbb{R}} \varphi_{AB}(x)\{x\} = AB = \sum_{x\in\mathbb{R}} \varphi_A(x)\{x\} \sum_{y\in\mathbb{R}} \varphi_B(y)\{y\}.$$

Since $\sum_{x\in\mathbb{R}} \varphi_A(x)\{x\} = \sum_{x\in\mathbb{R}} \varphi_A(x-y)\{x-y\}$, then,

$$\sum_{x\in\mathbb{R}} \varphi_{AB}(x)\{x\} = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y)\{x-y\}\{y\} = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y)\{x\}.$$

It follows that, at least formally,

$$\varphi_{AB}(x) = \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y).$$

Thus the product AB is mapped by φ to the convolution product $\varphi_A * \varphi_B$, where the convolution operator $\cdot * \cdot : (\Phi^{\mathbb{R}})^2 \to \Phi^{\mathbb{R}}$ is defined formally by

$$(\chi * \psi)(x) = \sum_{y\in\mathbb{R}} \chi(x-y)\psi(y).$$

If, for some $x \in \mathbb{R}$, the sum expressing $\varphi_{AB}(x)$ has infinitely many terms equal to $\{0\}$, then $\varphi_{AB}(x)$ is not determinate. In this case, φ_{AB} is determined, at best, on a proper subset of $\mathbb{R}$; therefore AB is not determinate. If A, B are to belong to an algebra $\mathbb{U}$, then, in particular, their product must be defined. This condition will be satisfied if we stipulate that, for each $A, B \in \mathbb{U}$ and each $x \in \mathbb{R}$, the sum expressing $\varphi_{AB}(x)$ shall have only finitely many non-empty terms. Since

$$\sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y) = \sum_{y\in B\cap[\inf B,\, x-\inf A]} \varphi_A(x-y),$$

there are at most as many such terms as there are elements in the set

$$B \cap [\inf B, x - \inf A],$$

which is finite if each non-empty $A \in \mathbb{U}$ satisfies these conditions:

(i) A is bounded below;
(ii) A has finite intersection with each finite interval.

An immediate consequence of these conditions is that each non-empty A contains its infimum. It turns out that (i) and (ii) are even sufficient to ensure that $\mathbb{U}$ is a division algebra. Thus we are led to

Definition 1.1. *The* ***real dyadic algebra*** $\mathbb{U}$ *is the set*

$$\{\varnothing\} \cup \{A \subset \mathbb{R} : \text{ for some } a_0 \in A,\ a_0 = \inf A \text{ and for each } \xi \in \mathbb{R},\ A \cap [a_0, \xi] \text{ is finite}\}$$

with the addition and multiplication defined by

$$\varphi_{A+B} = \varphi_A + \varphi_B, \quad \varphi_{AB} = \varphi_A * \varphi_B.$$

The binary operations in $\mathbb{U}$ are defined here in terms of characteristic functions rather than in terms of the corresponding elements of $\mathbb{U}$, as typified by A in the definition of the underlying set. It is not necessary, however, to appeal to the characteristic functions for such definitions: $A + B$ has been defined as $A \vartriangle B$; and

$$AB = \sum_{a\in\mathbb{R}} \varphi_A(a)\,\{a\} \sum_{b\in\mathbb{R}} \varphi_B(b)\,\{b\} = \sum_{a\in A}\sum_{b\in B} \{a+b\}\,.$$

The discussion preceding Definition 1.1 shows that, for each $A, B \in \mathbb{U}$, the product AB is defined (as also, of course, is $A + B$). In verifying that $\mathbb{U}$ is indeed an algebra, we shall next check that $A+B, AB \in \mathbb{U}$. If $A = B$, then $A+B = \varnothing \in \mathbb{U}$. We have $A + \varnothing = A \in \mathbb{U}$, $\varnothing + B = B \in \mathbb{U}$. If $\varnothing \neq A \neq B \neq \varnothing$, then

$$\inf(A + B) \geqq \inf\{\inf A, \inf B\}$$

and

$$\begin{aligned}(A + B) \cap [\inf(A + B), \xi] &\subseteqq (A \cup B) \cap [\inf(A + B), \xi] \\ &\subseteqq (A \cap [\inf A, \xi]) \cup (B \cap [\inf B, \xi])\,,\end{aligned}$$

which is finite; so $A+B \in \mathbb{U}$. If $A = \varnothing$ or $B = \varnothing$, then $AB = \varnothing \in \mathbb{U}$. If $A, B \neq \varnothing$, then

$$\inf AB = \inf A + \inf B,$$

and, with $a_0 := \inf A$, $b_0 := \inf B$,

$$AB \cap [\inf AB, \xi] \subseteqq [a_0 + b_0, \xi] \cap \sum_{a\in A\cap[a_0,\xi-b_0]} \ \sum_{b\in B\cap[b_0,\xi-a_0]} \{a + b\}\,,$$

which is finite; so $AB \in \mathbb{U}$. Thus addition and multiplication are closed in $\mathbb{U}$. This ensures that the following verifications of the remaining field axioms are not merely formal; in particular, summations formally over $\mathbb{R}$ are actually finite.

Addition and multiplication are commutative:

$$\varphi_{A+B} = \varphi_A + \varphi_B = \varphi_B + \varphi_A = \varphi_{B+A},$$

$$\varphi_{AB}(x) = \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y) = \sum_{y\in\mathbb{R}} \varphi_B(x-y)\varphi_A(y) = \varphi_{BA}(x);$$

and associative:

$$\varphi_{A+(B+C)} = \varphi_A + \varphi_{B+C} = \varphi_A + \varphi_B + \varphi_C = \varphi_{A+B} + \varphi_C = \varphi_{(A+B)+C},$$

$$\begin{aligned}\varphi_{A(BC)}(x) &= \sum_{y\in\mathbb{R}}\sum_{z\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y-z)\varphi_C(z) \\ &= \sum_{y\in\mathbb{R}}\sum_{z\in\mathbb{R}} \varphi_A(x-z-y)\varphi_B(y)\varphi_C(z) \\ &= \varphi_{(AB)C}(x);\end{aligned}$$

and multiplication is distributive over addition:

$$\begin{aligned}\varphi_{A(B+C)}(x) &= \sum_{y\in\mathbb{R}} \varphi_A(x-y)\left(\varphi_B(y)+\varphi_C(y)\right)\\ &= \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y) + \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_C(y)\\ &= \varphi_{AB+AC}(x).\end{aligned}$$

The zero of $\mathbb{U}$ is $\varnothing$, and the unity is $\{0\}$. Each $A \in \mathbb{U}$ is its own additive inverse; we shall not have occasion to use the expression $-A$. The existence of the multiplicative inverse $A^{-1} \in \mathbb{U}^*$ of each $A \in \mathbb{U}^* := \mathbb{U} - \{\varnothing\}$ may be proved thus:

If $A \in \mathbb{S}^* := \{S \in \mathbb{U}^* : \text{for some } x \in \mathbb{R},\ S = \{x\}\}$, then $A = \{\inf A\}$ and hence $A^{-1} = \{-\inf A\} \in \mathbb{S}^* \subset \mathbb{U}^*$. If $A \in \mathbb{U}^* - \mathbb{S}^*$, then we may write

$$A = \{a_0, a_1, \ldots\} \quad (a_0 < a_1 < \ldots)$$

and $A = \{a_0\}\, T$, where

$$T = \{a_0\}^{-1} A = \{-a_0\}\, A \in \mathbb{T} := \{T \in \mathbb{U}^* : \inf T = 0\}.$$

The notation assumes that A (and therefore T) is infinite, but it is not intended to exclude the case of finite A, T.

T^{-1} may be expressed as a (convergent) infinite product, using the notation of a *mixed product* $(0,\infty) \times \mathbb{U} \to \mathbb{U}$ defined by

$$xA = \{\xi : \text{ for some } \eta \in A,\ \xi = x\eta\}.$$

We shall show that

$$T^{-1} = \prod_{r=0}^{\infty} (2^r T) = \{0, t_1, \ldots\}\{0, 2t_1, \ldots\}\ldots \in \mathbb{T} \subset \mathbb{U}^*.$$

Convergence [1] of such an infinite product may be proved by using

Lemma 1.1. *A sequence (P_n) of subsets of $\mathbb{R}$ converges iff for some (C_n),*

$$C_n \subseteqq C_{n+1}\ (n \in \mathbb{P}), \quad \lim_{n\to\infty} C_n = \mathbb{R}, \quad C_n \cap P_\nu = C_n \cap P_n\ (\nu \geqq n).$$

If such a (C_n) exists, then $(C_n \cap P_n)$ converges, and

$$P := \lim P_n = \lim C_n \cap P_n.$$

Proof. If there is a (C_n) satisfying the proposed conditions, then

$$\bigcup_{n=1}^{\infty} C_n = \lim C_n = \mathbb{R}.$$

So, for each $x \in \mathbb{R}$, for some $m \in \mathbb{P}$, we have $x \in C_m$. Let $n(x)$ denote the least such m. Then $x \in C_{n(x)}$ and for each $\nu \geqq n(x)$,

$$\varphi_{P_\nu}(x) = \varphi_{C_{n(x)}\cap P_\nu}(x) = \varphi_{C_{n(x)}\cap P_{n(x)}}(x).$$

Thus the sequence $\left(\varphi_{P_\nu}\right)$ of characteristic functions converges to φ_P, defined by

$$\varphi_P(x) = \lim_{\nu\to\infty} \varphi_{P_\nu}(x) = \varphi_{C_{n(x)}\cap P_{n(x)}}(x).$$

Consequently (P_n) converges. $(C_n \cap P_n)$ also converges, for

$$C_n \cap P_n = C_n \cap P_{n+1} \subseteqq C_{n+1} \cap P_{n+1}.$$

It follows that

$$P = \mathbb{R} \cap P = \lim C_n \cap \lim P_n = \lim (C_n \cap P_n)\,.$$

Again, if (P_n) converges to P, then (φ_{P_n}) converges pointwise to φ_P, defined by

$$\varphi_P(x) = \lim_{\nu \to \infty} \varphi_{P_\nu}(x).$$

For each $n \in \mathbb{P}$, let

$$C_n := \{x : \text{ for each } \nu \geqq n,\ \varphi_{P_\nu}(x) = \varphi_{P_n}(x)\}\,.$$

Then $C_n \cap P_\nu = C_n \cap P_n$ $(\nu \geqq n)$; and $C_n \subseteqq C_{n+1}$, so that (C_n) converges to

$$C := \bigcup_{n=1}^{\infty} C_n \subseteqq \mathbb{R}.$$

Since (φ_{P_ν}) converges, for each $x \in \mathbb{R}$ there is an $n \in P$ such that $x \in C_n$. Hence $\mathbb{R} \subseteqq C$; so $\lim C_n = C = \mathbb{R}$. Thus (C_n) meets the proposed conditions. □

To prove that the infinite product expression of T^{-1} is convergent, we apply Lemma 1.1 to the sequence (P_n) of partial products, where

$$P_n := \prod_{r=0}^{n-1} (2^r T) = \{0, t_1, \ldots\} \ldots \{0, 2^{n-1} t_1, \ldots\}\,.$$

If the sequence (C_n) is defined by $C_n = (-\infty, 2^n t_1)$, then (C_n) satisfies the conditions of the lemma. Thus (P_n) converges and

$$P := \prod_{r=0}^{\infty} (2^r T) = \lim_{n \to \infty} P_n = \lim_{n \to \infty} \Big((-\infty, 2^n t_1) \cap \prod_{r=0}^{n-1} (2^r T) \Big).$$

To show that $P \in \mathbb{T}$, we note that $\inf P = 0$ and that

$$P \cap [0, \xi] \subseteqq P \cap C_{n(\xi)} = P_{n(\xi)} \cap C_{n(\xi)} = P_{n(\xi)} \cap [0, 2^{n(\xi)} t_1),$$

which is finite since $P_{n(\xi)} \in \mathbb{U}$. That $P = T^{-1}$ is seen by using the formula

$$T^{2^r} = 2^r T \quad (T \in \mathbb{T} := \{T \in \mathbb{U} : \inf T = 0\}\,,\ r \in \mathbb{Z})\,,$$

which may be proved by observing that

$$T^2 = \sum_{x \in T,\, y \in T,\, x<y} \{x + y\} + \sum_{x \in T,\, y \in T,\, x>y} \{x + y\} + \sum_{x = y \in T} \{x + y\}\,.$$

Renotation of (x, y) as (y, x) shows that

$$\sum_{x \in T,\, y \in T,\, x<y} \{x + y\} = \sum_{y \in T,\, x \in T,\, y<x} \{y + x\} = \sum_{x \in T,\, y \in T,\, x>y} \{x + y\}$$

and therefore

$$T^2 = \sum_{x = y \in T} \{x + y\} = \sum_{x \in T} \{2x\} = 2T.$$

It follows by induction that $T^{2^r} = 2^r T$ $(r \in \mathbb{N})$. This result is extended to negative r by using the left-associativity of the mixed product (an immediate consequence of the definition): thus, for each $r \in \mathbb{N}$,

$$T^{2^{-r}} = 2^{-r} 2^r T^{2^{-r}} = 2^{-r} T^{2^{-r} 2^r} = 2^{-r} T.$$

So $T^{2^r} = 2^r T$ $(r \in \mathbb{Z})$. Two applications of this equality yield

$$TP_n = T \prod_{r=0}^{n-1} (2^r T) = T \prod_{r=0}^{n-1} T^{2^r} = T^{2^n} = 2^n T = \{0, 2^n t_1, \ldots\}.$$

Hence, as in the proof of Lemma 1.1,

$$\varphi_{TP}(x) = \varphi_{C_{n(x)} \cap TP_{n(x)}}(x) = \varphi_{(-\infty, 2^{n(x)} t_1) \cap 2^{n(x)} T}(x) = \varphi_{\{0\}}(x),$$

which implies that $TP = \{0\}$, so that $T^{-1} = P$. It follows that

$$A^{-1} = \{a_0\}^{-1} T^{-1} = \{-a_0\} P = \{-a_0\} \prod_{r=0}^{\infty} (2^r (\{-a_0\} A)) \in \mathbb{U}.$$

We have proved that $\mathbb{U}$ is a field under the addition and multiplication proposed in Definition 1.1. Since Φ is a subfield of $\mathbb{U}$, the additive group $\mathbb{U}$, equipped with the restriction, to $\Phi \times \mathbb{U}$, of the multiplication in $\mathbb{U}$, is a vector space over Φ. For the same reason, this vector space, equipped with the multiplication in $\mathbb{U}$, is a commutative division algebra over Φ.

The conventional dyadic algebra $\mathbb{F}$ [3] is clearly isomorphic to the subalgebra

$$\mathbb{U}_{\mathbb{Z}} := \{A \in \mathbb{U} : A \subset \mathbb{Z}\}$$

of $\mathbb{U}$.

2. Rational Dyadic Exponentiation

To facilitate our treatment, in Section 3, of harmonic analysis in the subspace $\mathbb{U}_{\mathbb{Q}}$ of $\mathbb{U}$, we introduce, beyond addition and multiplication, a third operation:

Definition 2.1. ***Rational dyadic exponentiation*** *is the operation of raising a* ***base*** *$B \in \mathbb{U}$ to a* ***power*** *$B \uparrow X \in \mathbb{U}$, where the* ***exponent***

$$X \in \mathbb{U}_{\mathbb{Q}} := \{A \in \mathbb{U} : A \subset \mathbb{Q}\}.$$

The operator $\cdot \uparrow \cdot : \mathbb{U} \times \mathbb{U}_{\mathbb{Q}} \to \mathbb{U}$ is defined by

$$B \uparrow X = \sum_{x \in X} B^x$$

iff both the following conditions are satisfied:

(i) $B \neq \varnothing$ *or* $\inf X \geqq 0$;
(ii) X *is finite* *or* $\inf B > 0$.

Condition (i) excludes the occurrence of $\varnothing^x$ $(x < 0)$ as a term in the defining sum. Condition (ii) ensures convergence of $\sum_{x \in X} B^x$ to an element of $\mathbb{U}$.

It may be proved that for each

$$(B, x) \in \mathbb{U} \times \mathbb{Q} - \{\varnothing\} \times (-\infty, 0),$$

there is a unique $B^x \in \mathbb{U}$. It is not excluded that B^x takes one or more other values not in $\mathbb{U}$. Such failure of B^x to be single-valued does occur, but it is convenient to confine attention to the unique branch of this function taking values in $\mathbb{U}$. In this paper, then, the notation B^x will be understood to refer to this branch alone. We shall obtain an explicit expression (as a convergent infinite product) for the general rational power $B^{m/n}$ $(B \in \mathbb{U}, m \in \mathbb{Z}, n \in \mathbb{P}, (m,n) = 1)$. From this it will be clear that we have seized the intended branch of the function, that taking values in $\mathbb{U}$.

To reduce $B^{m/n}$ to a more convenient form, we define

$$T = \{-\inf B\}\, B \in \mathbb{T}.$$

Then $B = \{\inf B\}\, T$; so

$$B^{m/n} = \{\inf B\}^{m/n}\, T^{m/n} = \{(m/n)\inf B\}\, T^{m/n}.$$

A more manageable form of the exponent is

$$m/n = 2^k p/q,$$

where $k, p \in \mathbb{Z},\ q \in \mathbb{P}$, each of p, q is odd, and $(p,q) = 1$. These conditions determine k, p, q uniquely.

Using the canonical (monomorphic) embedding of the rational integers $\mathbb{Z}$ into the 2-adic integers $\mathbb{Z}_2$, and the fact [2] that the odd integers are invertible in $\mathbb{Z}_2$, we write $r := p/q$ as a 2-adic integer (in the notation of formal series)

$$\sum_{i\in\mathbb{N}} r_i 2^i = r = p/q = \sum_{i\in\mathbb{N}} p_i 2^i \Big/ \sum_{j\in\mathbb{N}} q_j 2^j.$$

Then, at least heuristically,

$$T^{p/q} = T^r = T^{\sum_{i\in\mathbb{N}} r_i 2^i} = \prod_{i\in\mathbb{N}} T^{r_i 2^i} = \prod_{i\in\mathbb{N}} (2^i T)^{r_i}.$$

Convergence of the infinite product to an element P of $\mathbb{U}$ is proved by writing

$$T = \{0, t_1, \ldots\} \quad (0 < t_1 < \ldots)$$

and applying Lemma 1.1, with $C_n = (-\infty, 2^n t_1)$, to the sequence of partial products

$$P_n := \prod_{i=0}^{n-1} (2^i T)^{r_i} = \{0, t_1, \ldots\}^{r_0} \ldots \{0, 2^{n-1} t_1, \ldots\}^{r_{n-1}},$$

as in the proof given earlier of the convergence of $T^{-1} = \prod_{i=0}^{\infty}(2^i T)$. Convergence of the infinite product P validates the assumption above that

$$T^{\sum_{i\in\mathbb{N}} r_i 2^i} = \prod_{i\in\mathbb{N}} T^{r_i 2^i}.$$

The formal series $r = p/q$ is computed by division of the series p by the series q, as in the following numerical illustration.

We shall calculate

$$B^{m/n} = \{1, 2, 8\}^{-3/10},$$

an easy example, since B is finite and has integer elements. We note first that

$$B^{m/n} = \{-3/10\}\, \{0, 1, 7\}^{2^{-1}(-3/5)} = \{-3/10\}\, (2^{-1}\, \{0, 1, 7\}^{-3/5})$$

and we evaluate $\{0, 1, 7\}^{-3/5}$. As a 2-adic integer,

$$-3 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + \ldots.$$

For brevity, we write this in *conventional* binary notation continued indefinitely *leftwards* from the (suppressed) binary point, with the leftwards recurring period $\ldots 1$ written as $\overline{1}$. In this notation, $-3 = \overline{1}01,\ 5 = \overline{0}101$, and the quotient

$$-3/5 = \overline{1}01/\overline{0}101 = \ldots 10011001 = \overline{1001}$$

is obtained by left-going long division, starting at the right. Hence

$$\{0,1,7\}^{-3/5} = \{0,1,7\}^{\overline{1001}} = \{0,1,7\}^{2^0}\,\{0,1,7\}^{2^3}\,\{0,1,7\}^{2^4}\ldots;$$

therefore

$$\begin{aligned}\{0,1,7\}^{-3/5} &= (2^0\,\{0,1,7\})(2^3\,\{0,1,7\})(2^4\,\{0,1,7\})\ldots\\ &= \{0,1,7\}\,\{0,8,56\}\,\{0,16,112\}\ldots\\ &= \{0,1,7,8,9,15,56,57,63\}\,\{0,16,112\}\ldots\\ &= \{0,1,7,8,9,15,16,17,23,24,25,31,\ldots\};\end{aligned}$$

so

$$\{1,2,8\}^{-3/10} = \{-3/10\}\,(2^{-1}\,\{0,1,7,8,9,15,16,17,23,24,25,31,\ldots\}).$$

The following properties of rational dyadic exponentiation $\cdot\uparrow\cdot$ will prove helpful in our development of harmonic analysis in $\mathbb{U}$:

Lemma 2.1. *For each $B \in \mathbb{U}^*$ $(\inf B > 0)$, the function $\beta_B : \mathbb{U}_{\mathbb{Q}} \to \mathbb{U}$ defined by*

$$\beta_B(X) = B \uparrow X$$

is a field monomorphism.

Proof. The conditions $B \in \mathbb{U}^*$, $\inf B > 0$ ensure that expressions entering this proof exist and converge. We omit the details. By an obvious generalisation of the equalities

$$\sum_{a\in X+Y}\{a\} = \sum_{x\in X}\{x\} + \sum_{y\in Y}\{y\}, \qquad \sum_{a\in XY}\{a\} = \sum_{x\in X}\sum_{y\in Y}\{x+y\},$$

we obtain

$$B\uparrow(X+Y) = \sum_{a\in X+Y} B^a = \sum_{x\in X} B^x + \sum_{y\in Y} B^y = B\uparrow X + B\uparrow Y,$$

$$B\uparrow(XY) = \sum_{a\in XY} B^a = \sum_{x\in X}\sum_{y\in Y} B^{x+y} = \sum_{x\in X} B^x \sum_{y\in Y} B^y = (B\uparrow X)\,(B\uparrow Y).$$

Thus β_B is a field homomorphism. To prove that β_B is injective, suppose that, for some $X, Y \in \mathbb{U}_{\mathbb{Q}}$, we have $X \neq Y$ and $\beta_B(X) = \beta_B(Y)$. Then

$$\beta_B(X+Y) = \beta_B(X) + \beta_B(Y) = \varnothing.$$

But, by our supposition, $X + Y \neq \varnothing$; so

$$\inf\,(\beta_B(X+Y)) = \inf\left(\textstyle\sum_{a\in X+Y} B^a\right) = (\inf B)\inf\,(X+Y).$$

Consequently $(\inf B)\inf\,(X+Y) \in \beta_B(X+Y)$; hence

$$\beta_B(X+Y) \neq \varnothing.$$

The contradiction shows that β_B is injective, and thus a field monomorphism. □

Lemma 2.2. *For each $B \in \mathbb{U}^*$, $X \in \mathbb{U}^*_{\mathbb{Q}} := \mathbb{U}_{\mathbb{Q}} - \{\varnothing\}$, $Y \in \mathbb{U}_{\mathbb{Q}}$ $(\inf B, \inf X > 0)$,*

$$B\uparrow(X\uparrow Y) = (B\uparrow X)\uparrow Y.$$

Proof. We prove first that for each $B \in \mathbb{U}^*$ $(\inf B > 0)$, $X \in \mathbb{U}^*_{\mathbb{Q}}$, $y \in \mathbb{Q}$,

$$B \uparrow (X^y) = (B \uparrow X)^y.$$

This is done by making the substitutions

$$y = p/q \ \ (p \in \mathbb{Z}, q \in \mathbb{P}, (p,q) = 1), \quad X = Z^q$$

and using Lemma 2.1:

$$\beta_B(X^y) = \beta_B(Z^p) = (\beta_B(Z))^p = ((\beta_B(Z))^q)^y = (\beta_B(Z^q))^y = (\beta_B(X))^y.$$

Equivalently, $B \uparrow (X^y) = (B \uparrow X)^y$. It follows that

$$(B \uparrow X) \uparrow Y = \sum_{y \in Y} (B \uparrow X)^y = \sum_{y \in Y} B \uparrow (X^y).$$

If Y is finite, then

$$(B \uparrow X) \uparrow Y = \sum_{y \in Y} B \uparrow (X^y) = B \uparrow \sum_{y \in Y} X^y = B \uparrow (X \uparrow Y)$$

and for this case a proof of associativity is complete. Suppose, then, that Y is infinite. Write $Y = \{y_0, y_1, \ldots\}$ $(y_0 < y_1 < \ldots)$ and we get

$$(B \uparrow X) \uparrow Y = \lim_{n \to \infty} \sum_{r=0}^{n-1} B \uparrow (X^{y_r}) = \lim_{n \to \infty} B \uparrow \sum_{r=0}^{n-1} X^{y_r}.$$

If Y is bounded above by some ξ, then $Y = Y \cap [\inf Y, \xi]$, which, by Definition 1.1, is finite, contrary to supposition. Thus Y is not bounded above; so $\lim y_n = \infty$. For some $n_0 \in \mathbb{N}$, then, for each $n > n_0$, we have $y_n > 0$ and may write

$$y_n = p_n/q_n \ \ (p_n, q_n \in \mathbb{P}, \ (p_n, q_n) = 1), \quad X = Z_n^{q_n}.$$

Hence

$$\inf X = q_n \inf Z_n, \quad X^{y_n} = Z_n^{q_n y_n} = Z_n^{p_n}.$$

So, since $X \in \mathbb{U}^*_{\mathbb{Q}}$,

$$\inf (X^{y_n}) = \inf (Z_n^{p_n}) = p_n \inf Z_n = y_n \inf X \in \mathbb{Q}, \ \inf (X^{y_n}) > 0.$$

Likewise

$$\inf \left(B^{\inf(X^{y_n})}\right) = \inf (X^{y_n}) \inf B = (y_n \inf X) \inf B.$$

Therefore

$$\inf A_n = \inf \sum_{x \in \sum_{r=n}^{\infty} X^{y_r}} B^x = \inf \left(B^{\inf(X^{y_n})}\right) = (\inf B)\, y_n \inf X.$$

Thus

$$\lim (\inf A_n) = \lim (\inf B)\, y_n \inf X = \infty,$$

which implies that $\lim A_n = \varnothing$. Consequently

$$(B \uparrow X) \uparrow Y = \lim_{n \to \infty} B \uparrow \sum_{r=0}^{n-1} X^{y_r} = \lim A_n + B \uparrow (X \uparrow Y) = B \uparrow (X \uparrow Y).$$

□

3. Fourier Analysis in the Rational Dyadic Space

We have seen that the additive group $\mathbb{U}$, equipped with the restriction to $\Phi \times \mathbb{U}$ of the multiplication in $\mathbb{U}$, is a vector space over Φ. We shall discuss harmonic analysis in the subspace $\mathbb{U}_{\mathbb{Q}}$ of $\mathbb{U}$, beginning with

Definition 3.1. *The* ***Fourier transform operator*** $\widehat{\cdot}: \mathbb{U}_{\mathbb{Q}} \to \mathbb{U}_{\mathbb{Q}}$ *is defined by*

$$\widehat{A} = \mathbb{P} \uparrow A = \sum_{x \in A} \mathbb{P}^x .$$

(The defining series exists and is convergent because $\mathbb{P} \neq \varnothing$ and $\inf \mathbb{P} > 0$. From now on we shall generally not mention such details, leaving them to the reader.)

This apparently arbitrary definition will be given some plausibility by a proof that $\widehat{\cdot}$ is a *self-inverse automorphism* of the field $\mathbb{U}_{\mathbb{Q}}$. (Our analysis differs in this respect from conventional Fourier analysis, whose transform operator is a *skew-inverse isomorphism* which preserves sums and takes pointwise products into convolution products and vice versa. As it were by way of compensation, *dyadic conjugation* is skew-inverse, not self-inverse like its conventional analogue, complex conjugation.)

A proof depends upon the following property of dyadic exponentiation $\cdot \uparrow \cdot$:

$$\mathbb{P} \uparrow \mathbb{P} = \{1\} .$$

To see this, we define $\mathbb{J} = \{0, 1\}$, and we sum the (convergent) geometric series

$$\sum_{r=0}^{\infty} \mathbb{P}^r = (\{0\} + \mathbb{P})^{-1} = \mathbb{N}^{-1} = \mathbb{J}$$

in the usual way. (The inverses quoted may be checked by multiplication.) Then

$$\mathbb{P} \uparrow \mathbb{P} = \sum_{x \in \mathbb{P}} \mathbb{P}^x = \sum_{r=0}^{\infty} \mathbb{P}^{r+1} = \mathbb{P} \sum_{r=0}^{\infty} \mathbb{P}^r = \mathbb{P}\mathbb{J} = \{1\} .$$

The element $\{1\}$ of $\mathbb{U}_{\mathbb{Q}}$ is a (left and right) *exponentiative identity*:

$$\{1\} \uparrow A = \sum_{x \in A} \{1\}^x = \sum_{x \in A} \{x\} = A, \qquad A \uparrow \{1\} = \sum_{x \in \{1\}} A^x = A^1 = A.$$

Using Lemma 2.2, we can prove that the Fourier transform $\widehat{\cdot}$ is self-inverse:

$$\widehat{\;}(\widehat{A}) = \mathbb{P} \uparrow (\mathbb{P} \uparrow A) = (\mathbb{P} \uparrow \mathbb{P}) \uparrow A = \{1\} \uparrow A = A.$$

Since, by definition, $\widehat{A} = \mathbb{P} \uparrow A = \beta_{\mathbb{P}}(A)$, Lemma 2.1 shows that $\widehat{\cdot}$ is monomorphic, and thus a self-inverse automorphism of $\mathbb{U}_{\mathbb{Q}}$.

It will be shown that the Fourier operator $\widehat{\cdot}$ may be viewed as an *orthogonal* transform. Such transforms are conventionally defined on a unitary space (a vector space over $\mathbb{C}$ with an Hermitian inner product). The definition of such an inner product uses complex conjugation. We therefore introduce an analogue $(\cdot)^*$ of the complex conjugation operator $\bar{\cdot}$.

The complex conjugate is already defined on the field $\mathbb{C}$ underlying a unitary space, and trivially induces conjugation of vectors in that space. The meagreness of the field Φ underlying the space $\mathbb{U}_{\mathbb{Q}}$ seems to preclude non-trivial definition of an analogous conjugate on Φ that would induce conjugation of vectors in $\mathbb{U}_{\mathbb{Q}}$. The operator $(\cdot)^*$ is therefore defined directly on (a subset of) $\mathbb{U}_{\mathbb{Q}}$:

Definition 3.2. *The* ***dyadic conjugate*** A^* *of* $A \in \mathbb{U}_{\mathbb{Q}}$ *is defined by*

$$\varphi_{A^*}(x) = \sum_{y \in \widehat{A}} \varphi_{\mathbb{P}^x}(y)$$

iff the summation on the right converges for each $x \in \mathbb{Q}$.

The dyadic conjugate A^* is not defined for every $A \in \mathbb{U}_{\mathbb{Q}}$. For example, $\{1\}^*$ is undefined. Indeed, $\widehat{\ }\{1\} = \mathbb{P} \uparrow \{1\} = \mathbb{P}$. So, formally,

$$\varphi_{\{1\}^*}(1) = \sum_{y \in \mathbb{P}} \varphi_{\mathbb{P}}(y) = \sum_{r=1}^{\infty} \{0\}\,.$$

But $(\cdot)^*$ is not empty. Indeed, $\widehat{\mathbb{P}} = \mathbb{P} \uparrow \mathbb{P} = \{1\}$. So, for each $x \in \mathbb{Q}$,

$$\varphi_{\mathbb{P}^*}(x) = \sum_{y \in \{1\}} \varphi_{\mathbb{P}^x}(y) = \varphi_{\mathbb{P}^x}(1)\,.$$

Definition 3.3. *The* ***inner product*** $\langle A, B\rangle \in \Phi$ *of* $A, B \in \mathbb{U}_{\mathbb{Q}}$ *is defined by*

$$\langle A, B\rangle = \sum_{x \in \mathbb{Q}} \varphi_A(x)\varphi_{B^*}(x)$$

iff B^* *is defined and the summation on the right converges.*

The definiens here is only formally a summation over $\mathbb{Q}$: it may be written

$$\sum_{x \in \mathbb{Q}} \varphi_A(x)\varphi_{B^*}(x) = \sum_{x \in A} \varphi_{B^*}(x) = \sum_{x \in B^*} \varphi_A(x).$$

The partial function $\langle\cdot,\cdot\rangle : \mathbb{U}^2 \to \Phi$ is symmetric: $\langle A, B\rangle = \langle B, A\rangle$. Indeed, if $\langle A, B\rangle$ is defined, then so is $B^* = \sum_{y \in \widehat{B}} \varphi_{\mathbb{P}^x}(y)$, which implies that this summation is finite. This justifies the last step in the calculation

$$\langle A, B\rangle = \sum_{x \in A} \varphi_{B^*}(x) = \sum_{x \in A}\sum_{y \in \widehat{B}} \varphi_{\mathbb{P}^x}(y) = \sum_{y \in \widehat{B}}\sum_{x \in A} \varphi_{\mathbb{P}^x}(y).$$

Since $\inf \mathbb{P}^x = x \inf \mathbb{P} = x$, we have $\varphi_{\mathbb{P}^x}(y) = \varnothing$ if $x > y$. So

$$\sum_{x \in A} \varphi_{\mathbb{P}^x}(y) = \sum_{x \in A \cap [\inf A, y]} \varphi_{\mathbb{P}^x}(y),$$

which shows that the summation is finite. By definition of addition, then,

$$\sum_{x \in A} \varphi_{\mathbb{P}^x}(y) = \varphi_{\sum_{x \in A} \mathbb{P}^x}(y) = \varphi_{\mathbb{P} \uparrow A}(y)\varphi_{\widehat{A}}(y).$$

Thus we obtain the $\mathbb{U}_{\mathbb{Q}}$-analogue of *Parseval's theorem:* iff $\langle A, B\rangle$ is defined,

$$\langle A, B\rangle = \sum_{y \in \widehat{B}} \varphi_{\widehat{A}}(y) = \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{B}}(y).$$

The symmetry of the last expression shows that $\langle A, B\rangle = \langle B, A\rangle$.

The function $\langle\cdot,\cdot\rangle$ is also bilinear: for each $\lambda, \mu \in \Phi$,

$$\langle A, \lambda B + \mu C\rangle = \lambda \langle A, B\rangle + \mu \langle A, C\rangle$$

iff $\langle A, B\rangle, \langle A, C\rangle$ are defined. Indeed, the condition mentioned implies that

$$\sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{B}}(y), \ \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{C}}(y)$$

are finite summations. So

$$\lambda \sum_{y\in\mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{B}}(y) + \mu \sum_{y\in\mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{C}}(y) = \sum_{y\in\mathbb{Q}} \varphi_{\widehat{A}}(y)\left(\lambda\varphi_{\widehat{B}}(y) + \mu\varphi_{\widehat{C}}(y)\right).$$

It is easy to see that, for each $A \in \mathbb{U}_{\mathbb{Q}}$, $\lambda\varphi_A(y) = \varphi_{\lambda A}(y)$. Hence

$$\lambda\varphi_{\widehat{B}}(y) + \mu\varphi_{\widehat{C}}(y) = \varphi_{\lambda\widehat{B}}(y) + \varphi_{\mu\widehat{C}}(y) = \varphi_{\lambda\widehat{B}+\mu\widehat{C}}(y) = \varphi_{\widehat{\ }(\lambda B+\mu C)}(y),$$

where the last equality follows from the automorphic property of $\widehat{\cdot}$. Therefore

$$\lambda\langle A, B\rangle + \mu\langle A, C\rangle = \sum_{y\in\mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{\ }(\lambda B+\mu C)}(y) = \langle A, \lambda B + \mu C\rangle.$$

Since Φ is not an ordered field, positive definiteness of the associated quadratic form is out of the question, so the properties of symmetry and bilinearity alone are sufficient justification for calling $\langle\cdot,\cdot\rangle$ an inner product.

As analogues of familiar properties of the conventional Fourier transform,

$$\widehat{\ }(\mathbb{P}^{\xi}) = \mathbb{P} \uparrow (\mathbb{P}^{\xi}) = (\mathbb{P} \uparrow \mathbb{P})^{\xi} = \{1\}^{\xi} = \{\xi\}$$

and, by the self-inverseness of $\widehat{\cdot}$,

$$\widehat{\ }\{\xi\} = \widehat{\ }(\widehat{\ }(\mathbb{P}^{\xi})) = \mathbb{P}^{\xi}.$$

To complete our view of $\widehat{\cdot}$ as an orthogonal transform, we prove

Theorem 3.1. *The set* $\{\mathbb{P}^{\xi} : \xi \in \mathbb{Q}\}$ *is orthonormal and complete in* $\mathbb{U}_{\mathbb{Q}}$.

Proof. The set $\{P^{\xi} : \xi \in \mathbb{Q}\}$ is orthonormal: for each $\xi, \eta \in \mathbb{Q}$ $(\xi \neq \eta)$,

$$\langle\mathbb{P}^{\xi}, P^{\eta}\rangle = \sum_{y\in\mathbb{Q}} \varphi_{\{\xi\}}(y)\varphi_{\{\eta\}}(y) = \sum_{y\in\{\xi\}} \varphi_{\{\eta\}}(y) = \varphi_{\{\eta\}}(\xi) = \varnothing,$$

$$\left\|\mathbb{P}^{\xi}\right\| = \langle\mathbb{P}^{\xi}, \mathbb{P}^{\xi}\rangle = \varphi_{\{\xi\}}(\xi) = \{0\}.$$

The orthonormal set $\{\mathbb{P}^{\xi} : \xi \in \mathbb{Q}\}$ is complete in $\mathbb{U}_{\mathbb{Q}}$: if, for each $A \in \mathbb{U}_{\mathbb{Q}}$, $\xi \in \mathbb{Q}$, we have $\langle A, \mathbb{P}^{\xi}\rangle = \varnothing$, then

$$\varnothing = \sum_{y\in\mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{\ }(\mathbb{P}^{\xi})}(y) = \sum_{y\in\mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\{\xi\}}(y) = \varphi_{\widehat{A}}(\xi).$$

Hence $\widehat{A} = \varnothing$, which implies that

$$A = \widehat{\ }(\widehat{A}) = \mathbb{P} \uparrow \widehat{A} = \mathbb{P} \uparrow \varnothing = \sum_{x\in\varnothing} \mathbb{P}^{x} = \varnothing.$$

□

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