

Chapter 5

DYADIC DISTRIBUTIONS

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Abstract We introduce the space $D'_d(R_+)$ of dyadic distributions and the space $S'_d(R_+)$ of dyadic tempered distributions based on the dyadic derivative.¹

1. Introduction

Distribution theory goes back to Sobolev [1], [2] and Schwartz [3]-[5], although the first generalized functions - for instance, the famous *Dirac delta-function* - appeared already in Dirac's papers on quantum mechanics. At present, distribution theory is well developed and is the subject of many monographs (see, for example, Gelfand and Shilov [6], Vekua [7], Antosik, Mikusinski and Sikorski [8], Zemanian [9], and other authors). Distributions have found a wide range of applications in mathematical physics, physics, quantum mechanics, and other branches of natural sciences (see, for example, Vladimirov [10], or Schwartz [3]). Distribution theory is used in the proof of some results of harmonic analysis (see, for instance, Stein [11], Ch. 3, Garcia-Cuerva and Rubio de Francia [12], Ch. 3, or Edwards [13], Ch. 3), the theory of functions of the complex variable [7], and other areas of mathematics.

In the monograph of Taibleson [14] a sketch of distribution theory on local fields is given. In the monograph of Vladimirov, Volovich and Zelenov [15] the theory of distributions on the field of p -adic numbers is developed.

Vilenkin [16] introduced the notion of the distribution on locally compact commutative group. But he restricted himself only by three concrete examples of such distributions. In particular, he did not introduce the operation of differentiation of these distributions.

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We introduce the space $D'_d(R_+)$ of *dyadic distributions* and the space $S'_d(R_+)$ of *dyadic tempered distributions* on the base of *dyadic derivative* which was introduced by Onneweer [17]. The completeness of the spaces $D'_d(R_+)$ and $S'_d(R_+)$ is proven.

2. Lemmas

Let N (or Z_+) be the set of all positive (respectively non negative) integers. For $x \in R_+$ and $n \in N$ we set

$$x_n \equiv [2^n x] \pmod{2}, \quad x_{-n} \equiv [2^{1-n} x] \pmod{2}, \quad (5.1)$$

where x_n and x_{-n} are equal to 0 or 1.

Let us introduce the distance $\rho^*(x, y)$ on R_+ as follows

$$\rho^*(x, y) = \sum_{i=1}^{+\infty} 2^{i-1} |x_{-i} - y_{-i}| + \sum_{i=1}^{+\infty} \frac{|x_i - y_i|}{2^i}, \quad x, y \in R_+.$$

It is not difficult to prove the inequalities

$$\rho(x, y) \equiv |y - x| \leq \rho^*(x, y), \quad x, y \in R_+.$$

The function $f : R_+ \rightarrow R$ is called uniformly ρ^* -continuous on R_+ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \rho^*(x, y) < \delta, \quad x, y \in R_+ \Rightarrow |f(y) - f(x)| < \varepsilon.$$

Let us consider the dyadic convolution $f * g$ of two functions $f \in L(R_+)$ and $g \in L^\infty(R_+)$:

$$(f * g)(x) = \int_{R_+} f(\rho^*(x, y)) g(y) dy.$$

LEMMA 5.1 *If the function f is uniformly ρ^* -continuous and bounded on R_+ , then for all $g \in R_+$, the dyadic convolution $(f * g)(x)$ is also uniformly ρ^* -continuous and bounded on R_+ .*

We define the function $h : (0, +\infty) \rightarrow (0, +\infty)$ by the equalities

$$h(x) = 2^n, \quad 2^n \leq x < 2^{n+1}, \quad n \in \mathbb{Z}.$$

Let us set

$$\Lambda_n^\alpha(x) = \int_0^{2^n} (h(t))^\alpha \psi(x, t) dt, \quad x \in R_+, \quad \alpha > 0.$$

LEMMA 5.2 For $\alpha > 0$, $n \in \mathbb{Z}_+$, the function Λ_n^α is bounded on R_+ and $\Lambda_n^\alpha \in L(R_+)$.

DEFINITION 5.1 Let $\alpha > 0$ and $f \in L(R_+) \cup L^\infty(R_+)$. If there exists the finite limit

$$f^{(\alpha)}(x) = \lim_{\alpha \rightarrow \infty} (f * \Lambda_n^\alpha)(x),$$

at the point $x \in R_+$, then the number $f^{(\alpha)}(x)$ is called the dyadic derivative (DD) of order α of the function f at the point x .

This definition has been introduced by C.W. Onneweer [17].

For $(x, y) \in R_+ \times R_+$, we set

$$t(x, y) = \sum_{n=1}^{\infty} (x_n y_{-n} + x_{-n} y_n).$$

This sum is finite, because $x_{-i} = 0$ for $i \geq i(x) \in \mathbb{N}$, (see (5.1)).

The generalized Walsh functions ψ_y ($y \in R_+$) are defined by the equalities

$$\psi_y(x) \equiv \psi(x, y) = (-1)^{t(x, y)}.$$

Let us consider the function system

$$\varphi_{m,n}(x) \equiv \psi(x, m2^{-n}) \mathbf{X}_{[0, 2^n)}(x), \quad (m \in \mathbb{N}, n \in \mathbb{Z}). \quad (5.2)$$

LEMMA 5.3 Each function of the system (5.2) is uniformly ρ^* -continuous on R_+ .

LEMMA 5.4 For each $\alpha > 0$ and $x \in R_+$ there exists the DD $\varphi_{m,n}^{(\alpha)}(x)$ and

$$\varphi_{m,n}^{(\alpha)}(x) = 2^{r\alpha} \varphi_{m,n}(x)$$

where the integer r is uniquely defined by the imbedding

$$[m2^{-n}, (m+1)2^{-n}) \subset [2^r, 2^{r+1}),$$

moreover

$$(\varphi_{m,n} * \Lambda_k^\alpha) = 2^{r\alpha} \varphi_{m,n} \quad \text{for } k \geq \max\{r, \log_2[(m+1)2^{-n}]\}.$$

3. The Space of Dyadic Distributions

DEFINITION 5.2 The function φ is called infinitely dyadic smooth on R_+ if there exists the dyadic derivative $\varphi^{(\alpha)}(x)$ for all $\alpha \in \mathbb{N}$, which is uniformly ρ^* -continuous and bounded on R_+ .

We denote by $C_W^{(\infty)}(R_+)$ the set of all such functions and by $\text{supp}\varphi$ the support of the function $\varphi : R_+ \rightarrow R$. The set

$$\Delta = [m2^{-n}, (m+1)2^{-n}), \quad m \in Z_+, \quad n \in Z,$$

is called *the dyadic interval*.

DEFINITION 5.3 *The function is called dyadic compactly supported, if:*

- 1 *There exists a dyadic interval Δ such that $\text{supp}\varphi^{(\alpha)} \subset \Delta \forall \alpha \in Z_+$,*
- 2 *For each $\alpha \in Z_+$ the sequence $(\varphi * \Lambda_n^{(\alpha)})(x)$ converges to $\varphi^{(\alpha)}(x)$ uniformly on R_+ .*

The set of all dyadic compactly supported functions from $C_w^{(\infty)}(R_+)$ will be denoted by $D_d(R_+)$. By the Lemmas 5.3 and 5.4 we have $D_d(R_+) \supset \varphi_{m,n}$ (see (5.2)).

REMARK 5.1 *There exists a compactly supported function $\varphi \in C_w^{(\infty)}(R_+)$ such that $\varphi \notin D_d(R_+)$. For example, $\varphi_{0,n}(x) = X_{[0,2^n)}(x)$, $(n \in Z)$ is such a function.*

DEFINITION 5.4 *The sequence $\{\varphi_n\}_{n=1}^{+\infty} \subset D_d(R_+)$ is called convergent in the space $D_d(R_+)$ to the function $\varphi_0 \in D_d(R_+)$ if:*

- 1 *There exists a dyadic interval Δ such that $\text{supp}\varphi_n^{(\alpha)} \subset \Delta$ for all $n, \alpha \in Z_+$;*
- 2 *$\varphi_n^{(\alpha)}(x)$ converges to $\varphi_0^{(\alpha)}(x)$ uniformly on Δ for each $\alpha \in Z_+$.*

In this case we will write: $\varphi_n \rightarrow \varphi_0$ in $D_d(R_+)$.

Let us consider the functional $f : D_d(R_+) \rightarrow R$. Its value on the element $\varphi \in D_d(R_+)$ will be denoted by (f, φ) .

DEFINITION 5.5 *The linear continuous functional $f : D_d(R_+) \rightarrow R$ is called dyadic distribution.*

We denote by $D'_d(R_+)$ the set of all dyadic distributions.

For any functions $f_1, f_2 \in D'_d(R_+)$ we set

$$(c_1 f_1 + c_2 f_2, \varphi) = c_1 (f_1, \varphi) + c_2 (f_2, \varphi), \quad \varphi \in D_d(R_+), \quad \forall c_1, c_2 \in R.$$

Then $D'_d(R_+)$ will be a linear space.

THEOREM 5.1 *Each function $f \in L_{loc}(R_+)$ generates a dyadic distribution, if we set*

$$(f, \varphi) = \int_{R_+} f(x)\varphi(x)dx, \quad \forall \varphi \in D_d(R_+). \quad (5.3)$$

Dyadic distributions of the form (5.3) are called *regular*. The remaining dyadic distributions are called *singular*.

An example of singular dyadic distribution gives *the dyadic δ -function*:

$$(\delta_d, \varphi) = \varphi(0), \quad \varphi \in D_d(R_+).$$

DEFINITION 5.6 *For $f \in D'_d(R_+)$ and $\alpha \in N$, we set*

$$(f^{(\alpha)}, \varphi) = (f, \varphi^{(\alpha)}), \quad \varphi \in D_d(R_+).$$

DEFINITION 5.7 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset D'_d(R_+)$ is called convergent to the dyadic distribution f_0 , if*

$$\lim_{n \rightarrow +\infty} (f_n, \varphi) = (f_0, \varphi), \quad \forall \varphi \in D_d(R_+).$$

In this case we will write: $f_n \rightarrow f_0$ in $D'_d(R_+)$.

DEFINITION 5.8 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset L_{loc}(R_+)$ is called to be convergent to the function $f_0 \in L_{loc}(R_+)$ in the space $L_{loc}(R_+)$, if this sequence converges to the function f_0 in the space $L(\Delta)$ for all dyadic intervals Δ .*

In this case we will write $f_n \rightarrow f_0$ in $L_{loc}(R_+)$.

THEOREM 5.2 *If $f_n \rightarrow f_0$ in the space $L_{loc}(R_+)$, then $f_n \rightarrow f_0$ in $D'_d(R_+)$.*

DEFINITION 5.9 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset D'_d(R_+)$ is called fundamental in $D'_d(R_+)$ if for each function $\varphi \in D_d(R_+)$ the sequence $\{(f_n, \varphi)\}_{n=1}^{+\infty}$ is fundamental in R .*

The following theorem states the completeness of the space $D'_d(R_+)$.

THEOREM 5.3 *If the sequence $\{f_n\}_{n=1}^{+\infty} \subset D'_d(R_+)$ is fundamental in $D'_d(R_+)$, then the functional defined by the equality*

$$(f, \varphi) = \lim_{n \rightarrow +\infty} (f_n, \varphi), \quad \forall \varphi \in D_d(R_+),$$

is linear and continuous on $D_d(R_+)$, e.g., $f \in D'_d(R_+)$.

4. The Space of Dyadic Tempered Distributions

Let us define the space of fast decreasing functions in the neighborhood of $+\infty$.

DEFINITION 5.10 *The function $\varphi \in C_w^{(+\infty)}(R_+)$ is said to belong to the set $S_d(R_+)$, if*

$$\lim_{x \rightarrow +\infty} x^\beta \varphi^{(\alpha)}(x) = 0, \quad \forall \alpha, \beta \in Z_+.$$

It is evident that the set $S_d(R_+)$ is a real linear space.

DEFINITION 5.11 *The sequence $\{\varphi_n\}_{n=1}^{+\infty} \subset S_d(R_+)$ is said to be convergent in the space $S_d(R_+)$ to the function $\varphi_0 \in S_d(R_+)$ if*

$$x^\beta \varphi_n^{(\alpha)}(x) \rightarrow x^\beta \varphi_0^{(\alpha)}(x), \quad n \rightarrow +\infty \quad \text{uniformly on } R_+ \quad \forall \alpha, \beta \in Z_+.$$

This fact will be written as follows: $\varphi_n \rightarrow \varphi_0$ in $S_d(R_+)$.

Let us consider the linear functional $f : S_d(R_+) \rightarrow R$. Its value on the function $\varphi \in S_d(R_+)$ will be denoted by (f, φ) . The linear functional $f : S_d(R_+) \rightarrow R$ is called *continuous* if the condition " $\varphi_n \rightarrow \varphi_0$ in $S_d(R_+)$ " implies

$$(f, \varphi_n) \rightarrow (f, \varphi), \quad \forall \varphi \in S_d(R_+).$$

DEFINITION 5.12 *The linear continuous functional $f : S_d(R_+) \rightarrow R$ is called the dyadic tempered distribution.*

The set of all dyadic tempered distributions will be denoted by $S'_d(R_+)$. Let us introduce the operations of addition of two dyadic tempered distributions $f_1, f_2 \in S'_d(R_+)$ and the multiplication of such distributions on real numbers as follows:

$$(c_1 f_1 + c_2 f_2, \varphi) = c_1 (f_1, \varphi) + c_2 (f_2, \varphi),$$

where $c_1, c_2 \in R_+$, and $\varphi \in S_d(R_+)$.

Then, the set $S'_d(R_+)$ is the real linear space which is conjugate to the space $S_d(R_+)$.

Let us recall the known terminology. The function $f : R_+ \rightarrow R$ has the polynomial growth in the neighbourhood of $+\infty$ if there exists a number $\beta \in Z$ such that $f(x) = O(x^\beta)$ as $x \rightarrow +\infty$.

THEOREM 5.4 *If a local integrable function $f(x)$ has the polynomial growth in the neighborhood of $+\infty$, then it generates the dyadic tempered distribution by the equality*

$$(f, \varphi) = \int_{R_+} f(x) \varphi(x) dx, \quad \varphi \in S_d(R_+).$$

Let us introduce the set of convergent sequences in the space $S'_d(R_+)$.

DEFINITION 5.13 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset S'_d(R_+)$ is said to be convergent to the element $f_0 \in S'_d(R_+)$, if*

$$\lim_{n \rightarrow +\infty} (f_n, \varphi) = (f_0, \varphi),$$

for each function $\varphi \in S_d(R_+)$.

DEFINITION 5.14 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset S'_d(R_+)$ is said to be fundamental in the space $S'_d(R_+)$, if the sequence $\{(f_n, \varphi)\}_{n=1}^{+\infty}$ is fundamental in R for each function $\varphi \in S_d(R_+)$.*

THEOREM 5.5 *The space $S'_d(R_+)$ is complete, i.e., if the sequence $\{f_n\}_{n=1}^{+\infty} \subset S'_d(R_+)$ is fundamental in the space $S'_d(R_+)$, then the functional f defined by the equality*

$$(f, \varphi) = \lim_{n \rightarrow +\infty} (f_n, \varphi), \quad \forall \varphi \in S_d(R_+),$$

belongs to the space $S'_d(R_+)$.

Let us define the operation of dyadic differentiation of a dyadic tempered distribution.

DEFINITION 5.15 *If $f \in S'_d(R_+)$ and $\alpha \in N$, then we set*

$$(f^{(\alpha)}, \varphi) = (f, \varphi^{(\alpha)}), \quad \varphi \in S_d(R_+).$$

It is evident that for every $\varphi \in S_d(R_+)$ we have $\varphi^{(\alpha)} \in S_d(R_+)$, $\forall \alpha \in N$. Therefore it follows from the Definition 5.15 that

$$\forall f \in S'_d(R_+) \quad \exists f^{(\alpha)} \in S'_d(R_+) \quad \forall \alpha \in N.$$

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