Chapter 6

ON DYADIC FRACTIONAL DERIVATIVES AND INTEGRALS

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Abstract The paper presents two dyadic analogs of the Lebesgue theorems on the differentiation of indefinite integral and on the integration of the derivative of a function. It is considered also the dyadic fractional integration by parts and the problem of dyadic fractional differentiation and integration of the integral depending on a real parameter. Most of these results are new also when considered dyadic derivatives and integrals of the first order. [1]

1. Introduction

The notions of pointwise and strong dyadic derivatives and integrals are known. The strong dyadic derivatives and integrals are defined on the segment $[0, 1]$ or on the positive half-line R_+ for some classes of functions. Sometimes they are defined on dyadic groups G or K, where G is isomorphic to the modified dyadic segment $[0, 1]^*$ and K is isomorphic to the modified half-line R_+^*.

P.L. Butzer and H.J. Wagner [1] introduced *the strong dyadic derivative* $D(f) \in L(G)$ for functions $f \in L(G)$. They proved that for the functions of Walsh-Paley system $\{w_n\}_{n=0}^{\infty}$ the equalities

$$D(w_n) = nw_n, \quad (n \in Z_+),$$

hold. This means that the Walsh-Paley functions are eigenfunctions of the operator D.

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In the same paper [1] for functions $f \in L(G)$ *the strong dyadic integral* $I(f) \in L(G)$ is introduced and proven are the equalities

$$D(T(f)) = I(D(f)) = f, \quad \text{if} \quad \hat{f}(0) \equiv \int_G f(x)d\mu(x) = 0,$$

where μ is the normalized Haar measure on G, i.e., $\mu(G) = 1$.

Thus, *the fundamental theorem of dyadic integral calculus* has been established. In the same paper the equivalence of the following three conditions was proven:

1 There exists $D(f) = g$;

2 There exists a function $g \in L(G)$ such that $\hat{g}(n) = n\hat{f}(n)$, $n \in Z$, where

$$\hat{f}(n) = \int_G f(x)w_n(x)d\mu(x),$$

are *Walsh - Fourier coefficients* of the function f;

3 There exists a function $g \in L(G)$ such that $f = I(G) + \hat{f}(0)$.

In another paper [2] by the same authors, the strong dyadic derivative $D(f)$ for functions $f \in L(R_+)$ is introduced and the equality $(D(f))^{\sim}(x) = x\tilde{f}(x)$ is proven, where $\tilde{f}$ is the Walsh - Fourier transformation of the function f.

For the functions $f \in L(R_+)$, H.J. Wagner [3] defined the strong dyadic integral $I(f)$ and proved the equalities

$$(D(I(f)) = f, \quad I(D(f)) = f.$$

(The last equality is proved under the condition $\tilde{f}(0) = 0$). In the same paper [3] the following criterion is proven: For a pair of functions $f, g \in L(R_+)$ the equality holds if and only if $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)/x$, $x > 0$.

In the paper [4], P.L. Butzer and H.J. Wagner investigated properties of *the pointwise dyadic derivatives* for functions defined on the segment $[0, 1]$.

C.W. Onneweer [5] introduced a *modified dyadic derivative* $D^{(1)}(f)$ for functions $f \in L(G)$ and proved the equalities

$$D^{(1)}(w_k) = 2^n w_k, \quad 2^n \leq k < 2^{n+1}, \quad n \in Z_+.$$

In the papers [5] and [6], C.W. Onneweer introduced also two strong dyadic derivatives $D^{(1)}(f)$ and $D^{[1]}(f)$ for functions $f \in L(K)$. In [6], he proved that the derivatives $D^{(1)}$ and $D^{[1]}$ have the same domain that does not coincide with $L(K)$. In the paper [7], C.W. Onneweer introduced *the pointwise p-adic derivative* $d_p^{(\alpha)}(f)(x)$ and *the strong p-adic derivative* $D^{(\alpha)}(f)$ *of fractional* *order* α for functions defined on the compact group G_p, which is the countable sum of cyclic groups of order $p \geq 2$. In this case, the strong p-adic derivative $D_p^{(\alpha)}(f)$ for functions in the spaces $L^q(G_p)$ and $C(G_p)$ is defined. C.W. Onneweer obtained the formulae for the derivatives of order α of functions that are the characters of the group G_p. It follows from this formulae that the characters of the group G_p are the eigenfunctions of the operator D_p^{α}.

In the same paper [7], *the strong* p*-adic integral of fractional positive order* α is introduced and the equalities

$$D_p^{(\alpha)}(I_p^{(\alpha)}(f)) = f, \quad I_p^{(\alpha)}(D_p^{(\alpha)}(f)) = f,$$

under condition $\int_{G_p} f(x)d\mu(x) = 0$ are proved. Also, a criterion is obtained in order that for the pair of functions f, g the equality $g = I_p^{(\alpha)}(f)$ or $g = D_p^{(\alpha)}(f)$ to be fulfilled.

Let us note that He Zelin [8] obtained the similar results for *the fractional derivatives and integrals of Butzer-Wagner type*.

The theory of dyadic differentiation and integration is presented explicitly in the books [9] and [10]. Some aspects of this theory are considered in the book [11].

In our paper [12] the modified strong dyadic integral $J_\alpha(f)$ and derivative $D^{(\alpha)}(f)$ of fractional positive order α have been introduced. Notice that the derivative $D^{(\alpha)}(f)$ for functions $f \in L(R_+)$ actually is a modification of the derivative of C.W. Onneweer, who considered the case $f \in L(K)$.

We formulate below two dyadic analogs of the Lebesgue theorems on the differentiation of indefinite integral and on the integration of the derivative of a function. We consider also dyadic fractional integration by parts and the problem of dyadic fractional differentiation and integration of the integral depending on a real parameter. The most results are new also for dyadic derivatives and integrals of the first order.

2. Definitions and Auxiliary Results

As usually, R_+ is the positive real axis. By $L^p(E)$, $1 \leq p \leq \infty$, we denote the space of Lebesgue measurable functions f on the measurable set $E \subset R_+$ with finite norm

$$\|f\|_{L_p(E)} = \left(\int_E |f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|f\|_{L^\infty(E)} = ess\sup_{x\in E}|f(x)|.$$

For a number $x \in R_+$ and a natural n, we set

$$x_n \equiv \lceil 2^n x \rceil \pmod{2}, \quad x_{-n} \equiv \lceil 2^{1-n} x \rceil \pmod{2}, \tag{6.1}$$

where $\lceil a \rceil$ is the integer part of the number a, x_n and x_{-n} are equal to 0 or 1.

Let us note that x_n (x_{-n}) is the n-th dyadic digit of the integer part (the fractional part respectively) of the number $x \in R_+$. The dyadic rational number $x \in R_+$ has finite dyadic expansion, i.e., $x_n = 0$ for $n \geq n(x)$.

As $x_{-n} = 0$ for $n \geq n(x)$, then for $(x, y) \in R_+ \times R_+$, the non negative integer

$$t(x, y) = \sum_{n=1}^{\infty} (x_n y_{-n} + x_{-n} y_n),$$

is defined.

Let us introduce *the Walsh kernel*

$$\psi(x, y) = (-1)^{t(x,y)}. \tag{6.2}$$

The Walsh - Fourier transform $F[f] \equiv \tilde{f}$ of the function $f \in L(R_+)$ is introduced by the equality

$$F[f](x) \equiv \tilde{f}(x) = \int_{R_+} \psi(x, y) f(y) dy. \tag{6.3}$$

This definition can be generalized for functions in the space $L^p(R_+)$, where $1 < p \leq 2$. In this case, the Walsh - Fourier transform $F[f] \equiv \tilde{f}$ is defined as the limit of the sequence

$$\int_0^{2^n} f(y)\psi(x, y) dy, \quad n \in Z_+,$$

by the norm of the space $L^p(R_+)$.

The properties of the Walsh - Fourier transform are similar to that of classical Fourier transform.

Let us introduce the operation of dyadic addition $\oplus$ on R_+ by setting

$$x \oplus y = z, \quad \text{for} \quad x, y \in R_0,$$

where the number z has dyadic digits $z_n = x_n + y_n \mod 2$, $n \in Z \setminus \{0\}$ and x_n, y_n are defined by the rule (6.1).

Let us note that

$$z_n = \sum_{n=1}^{\infty} 2^{n-1} z_{-n} + \sum_{n=1}^{\infty} \frac{z_n}{2^n},$$

and the case $z_n = 1$ for $n \geq n(z)$ is not excluded.

For the Walsh kernel (6.2) the equality

$$\psi(x \oplus y, t) = \psi(x, t)\psi(y, t), \tag{6.4}$$

holds, if $t, x, y \in R_+$ and $x \oplus y$ is not a dyadic rational (see [9], p. 421). Thus for fixed t and x the equality (6.4) for almost all $y \in R_+$ is valid.

Let us note that

$$\psi(x, k) = w_k(x - \lceil x \rceil), \quad (k \in Z_+, x \in R_+),$$

where $\{w_k\}_{k=0}^{\infty}$ is *the Walsh-Paley system*, which is orthonormal on the segment $[0, 1)$.

The Walsh-Fourier coefficients $\hat{f}(k)$ of the function $f \in L[0, 1)$ are defined by

$$\hat{f}(k) = \int_0^1 f(x) w_k(x) dx, \quad k \in Z_+. \tag{6.5}$$

Let us remind the definition of W-continuity of a function.

The function f is called W*-continuous* at the point $x \in R_+$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x \oplus y) - f(x)| < \varepsilon \quad \text{for} \quad 0 \leq y < \delta,$$

(see [11], Ch. 1).

Let us note that for each function $f \in L(R_+)$ its Walsh-Fourier transform $\tilde{f}$ is W-continuous on R_+ (see [11], theorem 6.1.5).

Let us introduce *the dyadic Lebesgue point* of a function.

The point $x \in R_+$ is called the dyadic Lebesgue point of a locally integrable function f, if this function is finite at this point and

$$\lim_{n \to +\infty} 2^n \int_0^{2^{-n}} |f(x \oplus t) - f(x)| dt = 0. \tag{6.6}$$

We generalize the notion of dyadic Lebesgue point as follows.

The dyadic segment of the rank n, $(n \in Z)$ is the set $\Delta_i^{(n)} = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ where $i \in Z_+$. The point $x \in R_+$ will be called *the binary d-point* of the function f, if $f(x)$ is finite and

$$\lim_{n \to +\infty} \frac{1}{|\Delta_i^{(n)}|} \int_{\Delta_i^{(n)}} f(t) dt = f(x), \tag{6.7}$$

where $\Delta_i^{(n)}$ is dyadic segment containing the point x and $|\Delta_i^{(n)}| = 2^{-n}$.

Every dyadic Lebesgue point of a function is also its binary d-point, i.e., (6.7) follows from (6.6).

For a locally Lebesgue integrable function f we set

$$F(x) = \int_0^x f(t) dt, \quad x \in R_+.$$

It is known that $F'(x) = f(x)$ almost everywhere on R_+ [15] and each point x for which the condition $F'(x) = f(x) \neq \pm\infty$ holds is a binary point of the function f (see [11], Ch. 1).

Thus, almost all points of the positive real axis R_+ are binary d-points of every function locally integrable on R_+. It can be proven that almost all points of the positive real axis r_+ are dyadic Lebesgue points of each function locally integrable on R_+.

For the function $f \in L[0,1)$, we set

$$S_m(f)(x) = \sum_{k=0}^{m-1} \hat{f}(k) w_k(x),$$

i.e., $S_m(f)(x)$ is *the Walsh-Fourier sum of order* m of the function f. The following theorem is known.

THEOREM 6.1 *Let the function $f \in L[0,1)$ has a finite value at the point $x \in [0,1)$. Then, the condition*

$$\lim_{n\to+\infty} S_{2^n}(f)(x) = f(x),$$

holds if and only if x is the binary d-point of the function f (see [11], p. 59).

This statement follows from the equality

$$S_{2^n}(f)(x) = \frac{1}{|\Delta_i^{(n)}|} \int_{\Delta_i^{(n)}} f(t)dt, \quad (n \in Z_+), \tag{6.8}$$

where $\Delta_i^{(n)}$ is the dyadic segment containing the point x (see [11], p. 45).

Let us formulate an analog of the Theorem 6.1 for the functions defined on R_+. For this aim we need *the generalized Walsh-Dirichlet integral* of the function f:

$$S_y(f)(x) = \int_0^y \tilde{f}(t)\psi(x,t)dt. \tag{6.9}$$

For the integral (6.9) the equality (6.8) is also valid (see [9], p. 428). Thus, the following theorem is true.

THEOREM 6.2 *Let the function $f \in L(R_+)$ has a finite value at the point $x \in [0,1)$ and $S_y(f)(x)$ is its generalized Walsh-Dirichlet integral. Then, the condition*

$$\lim_{n\to+\infty} S_{2^n}(f)(x) = f(x),$$

holds if and only if x is a binary d-point of the function f.

LEMMA 6.1 *For each $n \in Z$ and $\alpha > 0$ the function*

$$W_n^{\{\alpha\}}(x) \equiv \lim_{m\to+\infty} \int_{2^{-n}}^{2^n} \psi(x,y) y^{-\alpha} dy, \tag{6.10}$$

is well defined at each point $x > 0$, $W_n^{\{\alpha\}} \in L(R_+)$ and the limit in the right-hand side of (6.10) also exists in the metric of the space $L(R_+)$.

For $\alpha = 1$, the statement of this lemma is known (see [9], p. 434).

DEFINITION 6.1 *Let $\alpha > 0$, $f, g \in L^p(R_+)$, $1 \le p \le \infty$ and*

$$\lim_{n\to+\infty} \|f * W_n^{\{\alpha\}} - g\|_{L^p(R_+)} = 0.$$

Then, the function $g \equiv I_{\{\alpha\}}(f)$ is called the Wagner strong dyadic integral *(WSDI) of order α of the function f in the space $L^p(R_+)$.*

For $\alpha = 1$, $p = 1$, this definition was introduced by H.J. Wagner (see also [9], Ch. 9). Using this definition and 6.1 one can prove the following

THEOREM 6.3 *Let $\alpha > 0$, $f, g \in L(R_+)$. Then, the function g is the WSDI of order α of the function f in the space $L(R_+)$ if and only if*

$$\tilde{g}(0) = 0 \quad \textit{and} \quad \tilde{g}(x) = \tilde{f}(x) x^{-\alpha} \quad \textit{for} \quad x > 0.$$

For $\alpha = 1$ this theorem was proven by H.J. Wagner [3] (see also [9], p. 435).

3. Dyadic Analogs of Two Lebesgue Theorems

Let us remind the definition of dyadic convolution $f * g$ of two functions $f, g \in L[0,1)$:

$$(f * g)(x) = \int_{[0,1)} f(y) g(x \oplus y) dy, \quad x \in [0,1). \tag{6.11}$$

It is known that $f * g \in L(0,1)$ and $\widehat{(f * g)}(k) = \hat{f}(k)\hat{g}(k)$ where $\hat{f}$ are the Walsh - Fourier coefficients of the function f (see (6.5)).

The convolution (6.11) is well defined also for the case $f \in L^p(0,1)$, $1 \le p \le \infty$, $g \in L(0,1)$, and $f * g \in L^p[0,1)$.

Let us set

$$T_r^{(\alpha)}(x) = \sum_{k=0}^{2^r-1} k^\alpha w_k(x), \quad (\alpha \in R, r = 0, 1, \ldots). \tag{6.12}$$

In (6.12) and below we will assume $0^0 = 1$. We denote by $X[0,1)$ any of the spaces $L^p[0,1)$, $1 \le p \le \infty$.

He Zelin [8] introduced the following

DEFINITION 6.2 *Let $\alpha \in R$, $f \in X[0,1)$ and there exists a function $g \in X[0,1)$ such that*

$$\lim_{r\to\infty} \|f * T_r^{(\alpha)} - g\|_X = 0.$$

Then, for $\alpha > 0$ (or $\alpha < 0$) the function g is called the strong dyadic derivative (SDD) of order α (or the strong dyadic integral (SDI) of order $(-\alpha)$) of the function f in the space $X[0,1)$.

In both cases we will write $g = T^{(\alpha)} f$.

He Zelin [8] proved the equality $T^{(-1)} f = I(f)$, where $I(f)$ is the strong dyadic integral of the function $f \in X[0,1)$ in the space $X[0,1)$, which was introduced by P.L. Butzer and H.J. Wagner [1]. Moreover, in [8] for $\alpha \in R$, the equality

$$(\widehat{T^{(\alpha)} f})(k) = k^\alpha \hat{f}(k), \quad k \in Z_+, \quad \text{and} \quad f, T^{(\alpha)} f \in X[0,1),$$

has been proved.

For $\alpha > 0$ every function $f \in L^p[0,1)$ has the SDI $T^{(-\alpha)} f$ in the space $L^p[0,1)$ and

$$T^{(-\alpha)} f = (f * T_\infty^{(-\alpha)}),$$

where

$$T_\infty^{(\alpha)}(x) = \sum_{k=0}^{\infty} k^{-\alpha} w_k(x) \in L^p[0,1).$$

The series $\sum_{k=0}^{\infty} k^{-\alpha} w_k(x)$ converges to the function $T^{(-\alpha)} f$ in the space $L[0,1)$ and also at each point $x \in (0,1)$ if $\alpha > 0$.

It is proven in [8] that if $T^{(\alpha)} f$ exists in the space $L^p[0,1)$, $1 \le p \le \infty$ for some $\alpha \in R \setminus \{0\}$, then $T^{(-\alpha)}(T^{(\alpha)} f) = f$.

Let us introduce the pointwise analog of the Definition 6.2.

DEFINITION 6.3 *Let $\alpha \in R$, $f \in L[0,1)$ and there exists the finite limit*

$$t^{(\alpha)} f(x) \equiv \lim_{r\to\infty} (T_r^{(\alpha)} * f)(x)$$

at the point $x \in [0,1)$. Then, for $\alpha \ge 0$ (or $\alpha < 0$) the number $t^{(\alpha)} f(x)$ is called the dyadic derivative (DD) of order α (or the dyadic integral (DI) of order $(-\alpha)$ respectively) of the function f at the point x.

The following statement is true.

THEOREM 6.4 *Let $\alpha \in R \setminus \{0\}$ and the function $f \in L[0,1)$ has the SDD $T^{(\alpha)}f$ of order α in the space $L[0,1)$ for $\alpha > 0$ (or it has the SDI $T^{(\alpha)}f$ of order $(-\alpha)$ for $\alpha < 0$) in the space $L[0,1)$. Then, there exists $t^{(-\alpha)}(T^{(\alpha)}f)(x)$ at the point $x \in [0,1)$ if and only if x is a binary d-point of the function f. In this case,*

$$t^{(-\alpha)}(T^{(\alpha)}f)(x) = f(x).$$

In particular, the last equality is true almost everywhere on $[0,1)$.

It follows from this theorem the following corollary.

COROLLARY 6.1 *Under the conditions of the Theorem 6.4, the equality*

$$t^{(-\alpha)}(T^{(\alpha)}f)(x) = f(x)$$

holds at each dyadic Lebesgue point $x \in [0,1)$ of the function f. In particular, this equality is valid almost everywhere on $[0,1)$.

For $\alpha = -1$ this corollary may be considered as a dyadic analog of the classical Lebesgue theorem on the differentiation of indefinite Lebesgue integral. For $\alpha = 1$ it may be considered as a dyadic analog of the classical Lebesgue theorem on the integration of the ordinary derivative (see, for example, [16], Ch. 9).

Let us set

$$U_r^{(\alpha)}(x) = w_0(x) + \sum_{n=0}^{r} \sum_{j=0}^{2^n-1} 2^{n\alpha} w_{2^n+j}(x), \quad (\alpha \in R, r = 0, 1, \ldots).$$

DEFINITION 6.4 *Let $\alpha \in R$, $f \in X[0,1)$. If there exists a function $g \in X[0,1)$ such that*

$$\lim_{r \to \infty} \|f * U_r^{(\alpha)} - g\|_X = 0,$$

then for $\alpha > 0$ (or $\alpha < 0$) the function g is called the modified strong dyadic derivative (MSDD) of order α (or the modified strong dyadic integral (MSDI) of order $(-\alpha)$ respectively) of the function f in the space $X[0,1)$.

In this case we will write $g = U^{(\alpha)}f$.

DEFINITION 6.5 *Let $\alpha \in R$, $f \in L[0,1)$. If there exists the finite limit*

$$u^{(\alpha)}f(x) \equiv \lim_{r \to \infty} (U_r^{(\alpha)} * f)(x)$$

at the point $x \in [0,1)$*, then for* $\alpha > 0$ *(or* $\alpha < 0$*) the number* $u^{(\alpha)}f(x)$ *is called the modified dyadic derivative (MDD) of order* α *(or the modified dyadic integral (MDI) of order* $|\alpha|$ *respectively) of the function* f *at the point* x*.*

THEOREM 6.5 *Let* $\alpha \in R \setminus \{0\}$ *and for the function* $f \in L[0,1)$ *there exists* $U^{(\alpha)}f$ *in the space* $L[0,1)$*. Then there exists* $u^{(\alpha)}(U^{(\alpha)}f)(x)$ *at the point* $x \in [0,1)$ *if and only if* x *is a binary* d*-point of the function* f*. In this case*

$$u^{(-\alpha)}(U^{(\alpha)}f)(x) = f(x).$$

In particular this equality holds almost everywhere on $[0,1)$*.*

COROLLARY 6.2 *Let* $\alpha \in R \setminus \{0\}$ *and for the function* $f \in L[0,1)$ *there exists* $U^{(\alpha)}f$ *in the space* $L[0,1)$*. Then the equality*

$$u^{(-\alpha)}(U^{(\alpha)}f)(x) = f(x)$$

holds at each dyadic Lebesgue point $x \in [0,1)$ *of the function* f*, and hence almost everywhere on* $[0,1)$*.*

The results similar to the Corollary 6.2 for the functions $f \in L(R_+)$ were obtained by us in the paper [12] (see also [10], Ch. 3).

Let us note that for the functions $f \in L[0,1]$ the equality $(I_r(f))^{[r]}(x) = f(x)$ holds almost everywhere on $[0,1)$ if $\int_0^1 f(x)dx = 0$ and r is a natural number, where $I_r(f)$ is the strong dyadic integral and $f^{[r]}$ is the strong dyadic derivative of order r in the sense of Butzer and Wagner. That was proved by F. Shipp [17] as an answer on a question of H.J. Wagner [18]. But in the paper [17] there is not any characterization of the points in which the equality mentioned above holds.

Let us remind the notion of the dyadic convolution $f * g$ of two functions $f, g \in L(R_+)$

$$(f * g)(x) = \int_{R_+} f(y)g(x \oplus y)dy, \quad x \in R_+. \tag{6.13}$$

As for the ordinary convolution the following equality $(f \tilde{*} g) = \tilde{f}\tilde{g}$ holds. The dyadic convolution (6.13) is well defined also in the case $f \in L^p(R_+)$, $1 \le p \le \infty$, $g \in L(R_+)$, and $f * g \in L^p(R_+)$.

Let us introduce the notation

$$\Lambda_n^{\{\alpha\}}(x) = \int_0^{2^n} t^\alpha \psi(x,t)dt, \quad (x \in R_+, \alpha > 0, n \in Z).$$

This function is defined and bounded on R_+ for $\alpha > 0$ and $n \in Z$.

DEFINITION 6.6 *If $\alpha > 0$ and for the function $f \in L(R_+)$ there exists the finite limit*

$$d^{\{\alpha\}}(f)(x) = \lim_{n\to\infty}(f * \Lambda_n^{\{\alpha\}})(x)$$

at the point $x \in R_+$, then the number $d^{\{\alpha\}}(f)(x)$ is called the dyadic derivative (DD) of order α of the function f at the point x.

The following theorem is an analog of the Theorem 6.5 for functions defined on the positive half-line.

THEOREM 6.6 *Let $\alpha > 0$ and the function $f \in L(R_+)$ has the WSDI of order α in the space $L(R_+)$. Then the DD $d^{\{\alpha\}}(I_{\{\alpha\}}(f))(x)$ exists at the point $x \in R_+$ if and only if x is a binary d-point of the function f. In this case the equality $d^{\{\alpha\}}(I_{\{\alpha\}}(f))(x) = f(x)$ is valid. In particular, this equality holds almost everywhere on R_+.*

COROLLARY 6.3 *Let $\alpha \in R\backslash\{0\}$ and the function $f \in L(R_+)$ has the WSDI $I_{\{\alpha\}}(f)$ in the space $L(R_+)$. Then, the equality $d^{\{\alpha\}}(I_{\{\alpha\}}(f))(x) = f(x)$ holds at each dyadic Lebesgue point $x \in [0,1)$ of the function f, and hence almost everywhere on $[0,1)$. Therefore this equality is valid almost everywhere on the segment $[0,1)$.*

Let us note that the equality $(I_{\{1\}}(f))^{\{1\}}(x) = f(x)$ was proved by J. Pal and F. Shipp almost everywhere on R_+ (see [19] or [11], p. 445). But they used an other definition of the derivative, namely that one of P.L. Butzer and H.J. Wagner.

4. Dyadic Integration by Parts

The results of this section can be considered as the fractional dyadic analogs of classical formula of integration by parts.

THEOREM 6.7 *Let $f \in L^p[0,1)$, $g \in L^q[0,1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < \infty$ and for some $\alpha > 0$ the SDD $T^{(\alpha)}f$ and $T^{(\alpha)}g$ exist in the spaces $L^p[0,1)$ and $L^q[0,1)$ respectively. Then the equality*

$$\int_0^1 (T^{(\alpha)}f)(x)g(x)dx = \int_0^1 f(x)(T^{(\alpha)}g)(x)dx \tag{6.14}$$

holds. For $\alpha < 0$ the similar statement is valid also for the strong dyadic integrals $T^{(\alpha)}f$ and $T^{(\alpha)}g$.

An analog of this theorem is true also for the modified strong dyadic derivatives and integrals.

THEOREM 6.8 *Let $f \in L^p[0,1)$, $g \in L^q[0,1)$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \le p < \infty$, and for some $\alpha > 0$ the MSDD $U^{(\alpha)}f$ and $U^{(\alpha)}g$ exist in the spaces $L^p[0,1)$ and $L^q[0,1)$ respectively. Then the equality*

$$\int_0^1 (U^{(\alpha)}f)(x)g(x)dx = \int_0^1 f(x)(U^{(\alpha)}g)(x)dx,$$

holds. For $\alpha < 0$ the similar statement is valid also for the MSDI $U^{(\alpha)}f$ and $U^{(\alpha)}g$.

Let us formulate the similar results for the functions defined on the positive half-line R_+. To this aim we introduce the function $h : (0,+\infty) \to (0,+\infty)$ as follows

$$h(x) = 2^{-n}, \quad 2^n \le x < 2^{n+1}, \quad n \in Z. \tag{6.15}$$

For $\alpha > 0$ we set

$$\Lambda_n^\alpha(x) = \int_0^{2^n} (h(t))^{-\alpha}\psi(x,t)dt, \quad x \in R_+.$$

LEMMA 6.2 *For $\alpha > 0$, $n \in Z$, the function Λ_n^α is defined and bounded on the positive half-line R_+ and $\Lambda_n^\alpha \in L(R_+)$. Moreover,*

$$\tilde{\Lambda}_n^\alpha(x) = (h(x))^{-\alpha}X_{[0,2^n)}(x) \quad \textit{for} \quad x > 0 \quad \textit{and} \quad \tilde{\Lambda}_n^\alpha(0) = 0,$$

where X_E is the characteristic function of the set $E \subset R_+$ (see [10], Ch. 3).

Using this lemma we can introduce the following definition.

DEFINITION 6.7 *If for the function $f \in L^p(R_+)$, $1 \le p \le \infty$, there exists a function $\varphi \in L^p(R_+)$ such that*

$$\lim_{n\to+\infty} \|f * \Lambda_n^\alpha - \varphi\|_{L^p(R_+)} = 0,$$

then the function $\varphi \equiv D^{(\alpha)}f$ is called the modified strong dyadic derivative (MSDD) of order α of the function f in the space $L^p(R_+)$.

THEOREM 6.9 *Let $f \in L^p(R_+)$, $g \in L^q(R_+)$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \le p < \infty$ and for some $\alpha > 0$ the MSDD $D^\alpha f$ and $D^\alpha(g)$ exist in the spaces $L^p(R_+)$ and $L^q(R_+)$ respectively. Then the equality*

$$\int_{R_+} D^\alpha f(x)g(x)dx = \int_{R_+} f(x)D^\alpha g(x)dx$$

is valid.

LEMMA 6.3 *For $\alpha > 0$ and $n \in Z$, the function*

$$W_n^\alpha(x) \equiv \lim_{m\to\infty} \int_{2^{-n}}^{2^m} \psi(x,y)(h(y))^\alpha dy$$

at each point $x > 0$ is well defined, $W_n^\alpha \in L(R_+)$ and $W_n^\alpha(x) = 0$ for $x \geq 2^n$. (See [20] or [10], Ch. 3).

Using this lemma we can introduce the following definition.

DEFINITION 6.8 *Let $\alpha > 0$, $f, g \in L^p(R_+)$, $1 \leq p \leq \infty$, and*

$$\lim_{n\to+\infty} \|f * W_n^\alpha - g\|_{L^p(R_+)} = 0.$$

Then, the function $g \equiv J_\alpha f$ is called the modified strong dyadic integral (MSDI) of order α of the function f in the space $L^p(R_+)$.

THEOREM 6.10 *Let $f \in L^p(R_+)$, $g \in L^q(R_+)$ $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p < \infty$, $1 \leq q > \infty$, and for some $\alpha > 1$, the MSDI $J_\alpha f$ and $J_\alpha g$ exist in the spaces $L^p(R_+)$ and $L^q(R_+)$ respectively. Then the equality*

$$\int_{R_+} J_\alpha f(x) g(x) dx = \int_{R_+} f(x) J_\alpha g(x) dx$$

holds.

An analog of this theorem is valid also for the Wagner strong dyadic integrals.

THEOREM 6.11 *Let $f \in L^p(R_+)$, $g \in L(R_+)$, $frac1p + \frac{1}{q} = 1$, $1 \leq p < \infty$, $1 \leq q < \infty$, and for some $\alpha > 0$ the WSDI $I_{\{\alpha\}} f$ and $I_{\{\alpha\}} g$ exist in the spaces $L^p(R_+)$ and $L^q(R_+)$ respectively. Then, the equality*

$$\int_{R_+} I_{\{\alpha\}} f(x) g(x) dx = \int_{R_+} f(x) I_{\{\alpha\}} g(x) dx$$

holds.

5. Fractional Dyadic Integration and Differentiation of an Integral by a Parameter

Let us remind that according to the Lemma 6.2 for $\alpha > 0$ and $n \in Z_+$ the inclusion

$$\Lambda_n^\alpha \in L(R_+) \cap L^\infty(R_+)$$

is valid. Thus, we can introduce the following definition.

DEFINITION 6.9 *Let $\alpha > 0$ and for the function $f \in L(R_+) \cup L^\infty(R_+)$ there exists finite limit*

$$d^{(\alpha)}(f)(x) = \lim_{n\to+\infty} (f * \Lambda_n^\alpha)(x)$$

at the point $x \in R_+$.

Then the number $d^{(\alpha)}(f)(x) = (f)(x)$ is called the modified dyadic derivative (MDD) of order α of the function f at the point x.

Let us introduce the notation

$$d_n^{(\alpha)} f(x) = (f * \Lambda_n^\alpha)(x),$$

where $n \in Z_+$, $\alpha > 0$. Then the existence of MDD $d^{(\alpha)} f(x)$ of the function $f \in L(R_+) \cup L^\infty(R_+)$ at the point $x \in R_+$ can be written in the form

$$\lim_{n\to+\infty} d_n^{(\alpha)} f(x) = d^{(\alpha)} f(x).$$

Below for the function of two variables $f(x,t)$ the notation $d^{(\alpha)} f(x_0,t)$ denotes the MDD of order α at the point $x_0 \in R_+$ by the first argument.

Let us consider the problem of dyadic differentiation of the integral

$$F(x) = \int_E f(x,t)dt, \tag{6.16}$$

where E is a Lebesgue measurable set from R^m, $m \in N$ and $x \in R_+$.

THEOREM 6.12 *Let the Lebesgue measurable function $f : R_+ \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x,t)| \le \varphi(t)$ for all $x \in R_+$ and almost all $t \in E$. We also assume that there exists the MDD $d^{(\alpha)} f(x_0,t)$ at the point $x_0 \in R_+$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let the integral $\int_E d^{(\alpha)} f(x_0,t)dt$ converges and the sequence $d_n^{(\alpha)} f(x_0,t)$ has an integrable majorant on the set E. Then, the function (6.16) has the MDD of order α at the point x_0 and the equality*

$$d^{(\alpha)} F(x_0) = \int_E d^{(\alpha)} f(x_0,t)dt$$

is valid.

COROLLARY 6.4 *Let $\alpha > 0$, $\varphi \in L(R_+)$, and $h^{-\alpha}\varphi \in L(R)$. Then the Walsh-Fourier transform $\tilde{\varphi}$ of the function φ has the MDD $d^{(\alpha)}(\tilde{\varphi})(x)$ of order α at each point $x \in R_+$ and the equality*

$$d^{(\alpha)}(\tilde{\varphi})(x) = (h^{-\tilde{\alpha}}\varphi)(x)$$

holds.

The proof of this corollary is based on the Theorem 6.12 and the following lemma.

LEMMA 6.4 *The generalized Walsh function $\psi(\circ, y) \equiv \psi_y(\circ)$ has the MDD of order $\alpha > 0$ at each point $x \in R_+$. Moreover, $d^{(\alpha)}(\psi_0)(x) \equiv 0$ on R_+ and for $y > 0$ the equality*

$$d^{(\alpha)}(\psi_y)(x) \equiv (h(y))^{-\alpha}\psi_y(x)$$

is valid (see [10] , Ch. 3).

Let us note that the Lemma 6.4 was proved at first in our paper [20] and J. Pal [21] proved a similar result for pointwise derivative of first order of Butzer-Wagner type.

Now we consider the problem of fractional dyadic integration of the function (6.16). Taking into account the Lemma 6.3 we introduce the following definition.

DEFINITION 6.10 *Let $\alpha > 0$ and for the function $f \in L^\infty(R_+)$ there exists the finite limit*

$$j_\alpha(f)(x) = \lim_{n\to+\infty}(f * W_n^\alpha)(x)$$

at the point x. Then the number $j_\alpha(f)(x)$ is called the modified dyadic integral (MDI) of order α of the function f at the point x.

For the function $f \in L^\infty(R_+)$ we set

$$j_n^{(\alpha)} f(x) = (f * W_n^\alpha)(x)$$

where $n \in Z_+$, $\alpha > 0$. Then the existence of the MDI $j_n^\alpha f(x)$ for the function $f \in L^\infty(R_+)$ at the point $x \in R_+$ can be written in the form $\lim_{n\to+\infty} j_n^{(\alpha)} f(x) = j_\alpha f(x)$.

Below for the function of two variables $f(x,t)$ the symbol $j_\alpha f(x_0,t)$ denotes the MDI of order α at the point $x_0 \in R_+$ by the first argument.

THEOREM 6.13 *Let the Lebesgue measurable function $f : R_+ \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x,t)| \leq \varphi(t)$ for all $x \in R_+$ and almost all $t \in E$. We also assume that there exists the MDI $j_\alpha f(x_0,t)$ at the point $x_0 \in R_+$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let the integral $\int_E j_\alpha f(x_0,t)dt$ converges and the sequence $j_n^{(\alpha)} f(x_0,t)$ has an* *integrable majorant on the set E. Then, the function (6.16) has the MDI of order α at the point x_0 and the equality*

$$j_\alpha F(x_0) = \int_E j_\alpha f(x_0, t)dt,$$

is valid.

COROLLARY 6.5 *Let $\alpha > 0$, $\varphi \in L(R_+)$ and $h^\alpha \varphi \in L(R_+)$. Then, the Walsh-Fourier transform $\tilde{\varphi}$ of the function φ has the MDI $j_\alpha(\tilde{\varphi})(x)$ of order α at each point x and the equality*

$$j_\alpha(\tilde{\varphi})(x) = (h^{\tilde{\alpha}} \varphi)(x)$$

holds.

The proof of this corollary is based on the Theorem 6.13 and the following lemma.

LEMMA 6.5 *The generalized Walsh function $\psi(x, y) \equiv \psi_y(x)$ has the MDI of order $\alpha > 0$ at each point $x \in R_+$. Moreover, $j_\alpha(\psi_0)(x) \equiv 0$ on R_+ and for $y > 0$, the equality*

$$j_\alpha(\psi_y)(x) \equiv (h(y))^\alpha \psi_y(x)$$

is valid (see [20] or [10], Ch. 3).

Let us set

$$t_n^{(\alpha)} f(x_0, y) \equiv (T_n^{(\alpha)} * f(\circ, y))(x_0),$$

where the kernel $T_n^{(\alpha)}$ was introduced in (6.12).

THEOREM 6.14 *Let the Lebesgue measurable function $f : [0, 1) \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x, y)| \le \varphi$ for all $x \in [0, 1)$ and almost all $y \in E$. We also assume that there exists the MDD $t^{(\alpha)} f(x_0, y)$ for some $\alpha > 0$ (or the MDI $t^{(\alpha)} f(x_0, y)$ for some $\alpha < 0$) at the point $x_0 \in [0, 1)$ for almost all $y \in E$. Moreover, let the integral $\int_E t^{(\alpha)} f(x_0, y)dy$ converges and the sequence $t_n^{(\alpha)} f(x_0, y)$ has an integrable majorant on the set E. Then, the function (6.16) has the DD $t^{(\alpha)} F(x_0)$ of order $\alpha > 0$ (or the DI $t^{(\alpha)} F(x_0)$ of order $(-\alpha)$) at the point x_0 and the equality*

$$t^{(\alpha)} F(x_0) = \int_E t^{8\alpha)} f(x_0, y)dy$$

is valid.

Using the kernel $W_n^{\{\alpha\}}(x)$ defined in Lemma 6.1, we introduce the following definition.

DEFINITION 6.11 *If $\alpha > 0$ and there exists the finite limit*

$$i_{\{\alpha\}}(f)(x) = \lim_{n\to+\infty}(f * W_n^{\{\alpha\}})(x)$$

for the function $f \in L^\infty(R_+)$ at the point $x \in R_+$, then the number $i_{\{\alpha\}}(f)(x)$ is called the Butzer-Wagner dyadic integral (BWDI) of order α of the function f at the point x.

Let us set

$$i_n^{\{\alpha\}}f(x,t) = \int_{R_+} f(y,t)W_n^{\{\alpha\}}(x \oplus y)dy, \quad x \in R_+, t \in E. \tag{6.17}$$

THEOREM 6.15 *Let the Lebesgue measurable function $f : R_+ \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x,t)| \leq \varphi(t)$ for all $x \in R_+$ and $t \in E$. We also assume that there exists BWDI $i_{\{\alpha\}}f(x_0,t)$ at the point $x_0 \in R_+$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let the integral $\int_E i_{\{\alpha\}}f(x_0,t)dt$ converges and the sequence $i_n^{\{\alpha\}}f(x_0,t)$ has an integrable majorant on the set E. Then, the function (6.16) has the BWDI $i_{\{\alpha\}}F(x_0)$ of order α at the point x_0 and the equality*

$$i_{\{\alpha\}}F(x_0) = \int_E i_{\{\alpha\}}f(x_0,t)dt$$

is valid.

Now let us consider the problem of modified strong dyadic differentiation and integration of the function (6.16). We will use the notation

$$d_n^\alpha f(x,t) = \int_{R_+} f(y,t)\Lambda_n^\alpha(x \oplus y)dy, \quad x \in R_+, t \in E, \alpha > 0.$$

THEOREM 6.16 *Let the measurable function $f : R_+ \times E \to R$ is such that*

$$\int_{R_+}\left\{\int_E |f(x,t)|dt\right\}dx < \infty, \tag{6.18}$$

and there exists the MSDD $D^\alpha f(x,t)$ by the parameter x in the space $L(R_+)$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let

$$\lim_{n\to\infty}\int_E \|d_n^\alpha f(\circ,t) - D^\alpha f(\circ,t)\|_{L(R_+}dt = 0.$$

Then the function (6.16) has the MSDD $D^\alpha F$ of order α in the space $L(R_+)$ and

$$D^\alpha F(\circ) = \int_E D^\alpha f(\circ, t)dt.$$

Let us introduce the notation

$$j_n^\alpha f(x,t) = \int_{R_+} f(y,t)W_n^\alpha(x \oplus y)dy, \quad x \in R_+, t \in E.$$

THEOREM 6.17 *Let the measurable function $f : R_+ \times E \to R$ satisfies the condition (6.18). Moreover, let the function $f(x,t)$ has MSDI $J_\alpha f(\circ, t)$ of some order $\alpha > 0$ in the space $L(R_+)$ for almost all $t \in E$ and*

$$\lim_{n\to\infty} \int_E \|j_n^\alpha f(\circ, t) - J_\alpha f(\circ, t)\|_{L(R_+)} dt = 0.$$

Then the function (6.16) has MSDI $J_\alpha F$ of order α in the space $L(R_+)$ and

$$J_\alpha f(\circ) = \int_E J_\alpha f(\circ, t)dt.$$

The similar result is valid also for the Wagner strong dyadic integral.

THEOREM 6.18 *Let the measurable function $f : R_+ \times E \to R$ satisfies the condition (6.18). Moreover, let the function $f(x,t)$ has the WSDI $I_{\{\alpha\}} f(\circ, t)$ of some order $\alpha > 0$ in the space $L(R_+)$ for almost all $t \in E$ and*

$$\lim_{n\to\infty} \int_E \|i_n^{\{\alpha\}} f(\circ, t) - I_{\{\alpha\}} f(\circ, t)\|_{L(R_+)} dt = 0.$$

where $i_n^{\{\alpha\}} f(x,t)$ is defined in (6.17). Then the function (6.16) has the WSDI $I_{\{\alpha\}} F$ of order α in the space $L(R_+)$ and the equality

$$I_{\{\alpha\}} F(\circ) = \int_E I_{\{\alpha\}} f(\circ, t)dt$$

holds.

Let us introduce the notation $t_r^{(\alpha)} f(\circ, t) = T_r^{(\alpha)} * f(\circ, t)$, where the kernel $T_r^{(\alpha)}$ is defined by the equality (6.12).

THEOREM 6.19 *Let the measurable function $f : R_+ \times E \to R$ satisfies the condition*

$$\int_{[0,1)} \{|f(x,t)|dt\} \, dx < \infty.$$

Moreover, let the function $f(x,t)$ has the SDD $T^{(\alpha)}f(\circ,t)$ of some order $\alpha > 0$ (or it has the SDI $T^{(\alpha)}f(\circ,t)$ if $\alpha < 0$) for almost all $t \in E$ and

$$\lim_{n\to\infty} \int_E \|t_n^\alpha f(\circ,t) - T^{(\alpha)}f(\circ,t)\|_{L[0,1)}dt = 0.$$

Then the function (6.16) has the SDD $T^{(\alpha)}F$, if $\alpha > 0$ (or it has the SDI $T^{(\alpha)}F$, if $\alpha < 0$) in the space $L[0,1)$ and the equality

$$T^{(\alpha)}F(\circ) = \int_E T^{(\alpha)}f(\circ,t)dt$$

is valid.

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