Chapter 10

DISCRETE-TYPE RIESZ PRODUCTS

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Abstract We factorize finite data of length m, or step functions determined on the intervals $[k/m,(k+1)/m), k=0,\ldots,m-1$ of $[0,1)$, by writting them as a discrete Riesz-type Product $t_n=\prod_{k=1}^{m}(1+a_k h_{k,n})$ with respect to the rows h_k of a matrix $H(m)$ of order $m\times m$ and associated to a sequence of coefficients $\{a_k : k=1,\ldots,m\}$. We give sufficient conditions on $H(m)$ and $\{a_k\}$, providing invertibility of the underlying non-linear Riesz-type transform and we present examples of classes of acceptable matrices.

1. Introduction

The original Riesz's construction associated to a sequence of coefficients $\{a_n\}$, was to show that there exists a continuous function F of bounded variation in $[0,2\pi)$, whose Fourier-Stieltjes coefficients do not vanish at infinity, F being the pointwise limit of the sequence of functions:

$$F_N(x)=\int_0^x \prod_{n=1}^{N}(1+a_n\cos(2\pi 4^n t))dt.$$

Over the years, Riesz's construction was generalized, by replacing the generating function $cos(2\pi t)$ with other generating functions such as the Rademacher, or Walsh functions, or trigonometric polynomials (see [4], [6], [6]). Recently in [3], multiscale Riesz Products have been constructed, based on a real valued function H on $[0, 1)$, called generating function and a dilation operator $T : [0, 1) \to [0, 1)$, such that:

$$\mu_m(\gamma) = \prod_{n=1}^{m} (1 + a_n H(T^{n-1}\gamma))$$

converges weak-$*$ to a bounded measure as $m \to \infty$. Obviously, we can generalize the definition of μ_m, by considering partial Riesz Products of the form:

$$\mu_m(\gamma) = \prod_{n=1}^{m} (1 + a_n H_n(\gamma)), \tag{10.1}$$

where $H_n(\gamma), (n = 1, \ldots, m)$ are bounded functions on $[0, 1)$. Clearly, if we denote by V_m the space of sequences of length m and by $B[0, 1)$ the space of bounded functions on $[0, 1)$, the partial Riesz Products (10.1) induce a non-linear transform $\mu_m : V_m \to B[0, 1)$, such that for every $a = \{a_1, \ldots, a_m\} \in V_m$ we have:

$$\mu_m(a)(\gamma) = \prod_{n=1}^{m} (1 + a_n H_n(\gamma)).$$

In order to achieve invertibility for μ_m, in [2] and [3] we considered step functions H_n on the intervals $\Omega_{n,m} = \left[\frac{n-1}{m}, \frac{n}{m}\right)$, $n = 1, \ldots, m$:

$$H_n(\gamma) = \sum_{i=0}^{m} h_{n,i} \mathbf{1}_{\Omega_{\mathrm{i,m}}}(\gamma).$$

As a consequence, we dealt with discrete Riesz-type products of the form:

$$t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}). \tag{10.2}$$

We proved the following:

THEOREM 10.1 *(see [3])*
Let $H(m) = \{h_{k,n} : k, n = 1, \ldots, m\}$ be a real orthonormal matrix whose first row is the constant row $(\frac{1}{\sqrt{m}}, \ldots, \frac{1}{\sqrt{m}})$ and all rows satisfy

$$h_n h_l = h_{n,l_0} h_l \quad \textit{whenever } n < l \tag{10.3}$$

where h_n, h_l are rows of $H(m)$ and h_{n,l_0} is the first non-zero entry of the l-row of the matrix $H(m)$. If $t = \{t_1, \ldots, t_m\}$ is a sequence of real numbers such that

$$\langle t, h_i \rangle \neq 0, \ i = 1, \ldots, m,$$

where $<,>$ *is the usual inner product of* $\mathbf{R}^m$*, then there is a unique sequence of coefficients* $\{a_k : k = 1, \ldots, m\}$ *such that:*

$$t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}). \tag{10.4}$$

Moreover, the coefficients $\{a_n : n = 1, \ldots, m\}$ are computed via the following:

$$a_n = \begin{cases} \langle t, h_1 \rangle - \sqrt{m} & n = 1 \\ \frac{\langle t, h_n \rangle}{\prod_{k=1}^{n-1}(1 + a_k h_{k,n_0})}, & n = 2, \ldots, m \end{cases},$$

where h_{n,n_0} is the first non-zero entry of the row h_n.

Also, we constructed a class of unbalanced Haar matrices $H(m)$ satisfying (10.3) of Theorem 10.1. An example is shown below:

$$H(3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$H(6) = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

In this paper, we relax the conditions imposed on the matrix $H(m)$ in Theorem 10.1. In Section 2, we see that Theorem 10.1 is true, if orthonormality is replaced by invertibility. Also, we show that for a particular class of exponential matrices we can drop (10.3) and Theorem10.1 is valid, as long as the values of the coefficients $\{a_k\}$ are restricted to the discrete set $A = \{0, 1\}$.

2. Discrete Riesz Products

In this section we obtain classes of matrices, whose corresponding Riesz Products give rise to an invertible non-linear transform.

PROPOSITION 10.1 *Let* $H(m) = \{h_{k,n} : k, n = 1, \ldots, m\}$ *be a real invertible matrix satisfying (10.3) of Theorem 10.1. If* $t = \{t_1, \ldots, t_m\}$ *is a sequence of real numbers such that*

$$\langle t, h_i \rangle \neq 0, \ i = 1, ..., m,$$

then there is a unique sequence of coefficients $\{a_k : k = 1, \ldots, m\}$ *such that:*

$$t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}).$$

Moreover, the coefficients $\{a_n : n = 1, \ldots, m\}$ *are computed via the following:*

$$a_n = \begin{cases} \langle t, h^{-1}_{\cdot,1}\rangle - \langle 1, h^{-1}_{\cdot,1}\rangle & n = 1 \\ \frac{\langle t, h^{-1}_{\cdot,n}\rangle}{\prod_{k=1}^{n-1}(1+a_k h_{k,n_0})}, & n = 2, \ldots, m \end{cases},$$

where $H^{-1}(m) = [h^{-1}_{j,k}]$ *is the inverse matrix of* $H(m)$*.*

Proof. We expand the discrete Riesz Product and we use (10.3) to get:

$$\begin{aligned} t_n &= 1 + \sum_{k=1}^{m} a_k h_{k,n} + \sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} a_{k_1} a_{k_2} h_{k_1,k_2^0} h_{k_2,n} + \ldots \\ &\quad + (a_1 \ldots a_m) \left(\prod_{j=1}^{m-1} h_{k_j,k_m^0} \right) h_{m,n}, \end{aligned}$$

where h_{k_j,k_m^0} is the first non-zero entry of the row h_{k_j}.

The invertibility of $H(m)$ and (10.4) imply that $\left\langle t, h^{-1}_{\cdot,1} \right\rangle = \left\langle 1, h^{-1}_{\cdot,1} \right\rangle + a_1$. For any $s > 1$ we have:

$$\begin{aligned} \left\langle t, h^{-1}_{\cdot,s} \right\rangle &= a_s \left(1 + \sum_{k_1=1}^{s-1} a_{k_1} h_{k_1,s_0} + \sum_{k_1=1}^{m-2} \sum_{k_2=k_1+1}^{m-1} a_{k_1} a_{k_2} \left(\prod_{j=1}^{2} h_{k_j,s_0} \right) \right. \\ &\quad \left. + \ldots + (a_1 \ldots a_{s-1}) \left(\prod_{j=1}^{s-1} h_{k_j,s_0} \right) \right) \\ &= a_s \prod_{k=1}^{s-1} (1 + a_k h_{k,s_0}). \end{aligned}$$

EXAMPLE 10.1 *A class of matrices* $H(m)$ *satisfying Proposition 10.1 is produced by the following rules:*

(a) The first row of $H(m)$ *is the constant row* $\{1, \ldots, 1\}$*.*

(b) Every other row has only two non-zero entries 0 *or* 1*.*

(c) *If we denote by supp*$\{h_k\} = \{j \in \{1,\ldots,m\} : h_{kj} \neq 0\}$, *then:*

$$supp\{h_k\} \cap supp\{h_l\} = \oslash \quad or\ supp\{h_k\} \cap supp\{h_l\} = supp\{h_l\}$$

whenever $k < l$.

Below, we present examples of matrices satisfying rules (a)-(c):

$$\begin{aligned} H(3) &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ H(6) &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

PROPOSITION 10.2 *Let* $\Theta(m) = \{\theta_{n,j} : |\theta_{n,j}| < \pi,\ n, j = 1,\ldots,m\}$ *be an invertible matrix whose columns satisfy the following:*

$$-\pi \leq \sum_{n=1}^{m} \theta_{n,j} \leq \pi,\ j = 1,...,m.$$

If $t = \{t_j = |t_j|e^{i\arg(t_j)}, j = 1,...,m\}$ $-\pi \leq arg(z) \leq \pi$ *is a sequence of complex numbers, then there is a unique sequence of boolean coefficients* $\{a_n : n = 1,\ldots,m\}$, *such that:*

$$t_j = \prod_{n=1}^{m} (1 + a_n e^{i\theta_{n,j}}).$$

Moreover, the coefficients $\{a_n : n = 1,\ldots,m\}$ *are computed via the following matrix equation:*

$$a = 2\Theta^{-1}C(t),$$

where $a = [a_n]$ *and* $C(t) = [\arg(t_n)]$ *are column matrices of order* $m \times 1$.

Proof. Let $t_j = \prod_{n=1}^{m}(1 + a_n e^{i\theta_{n,j}})$, where $a_n \in \{0,1\}$, then we have:

$$t_j = \prod_{n=1}^{m} (1 + a_n e^{i\theta_{n,j}}) = \prod_{n=1}^{m} (1 + e^{i\theta_{n,j}})^{a_n}.$$

Since

$$\overline{t_j} = \prod_{n=1}^{m} (1 + e^{-i\theta_{n,j}})^{a_n} = \prod_{n=1}^{m} \left(e^{-i\theta_{n,j}} \left(e^{i\theta_{n,j}} + 1\right)\right)^{a_n}$$

$$= \prod_{n=1}^{m} \left(e^{-ia_n\theta_{n,j}} \left(1 + e^{i\theta_{n,j}}\right)^{a_n} \right) = t_j e^{-i \sum_{n=1}^{m} a_n \theta_{n,j}},$$

we get:

$$e^{-i \arg(t_j)} = e^{i \arg(t_j)} e^{-i \sum_{n=1}^{m} a_n \theta_{n,j}},$$

thus:

$$\sum_{n=1}^{m} a_n \theta_{n,j} = 2 \arg(t_j) + 2\lambda_j \pi, \lambda \in \mathbf{Z}.$$

The hypothesis $-\pi \leq \sum_{n=1}^{m} \theta_{n,j} \leq \pi$ indicates that $\lambda_j = 0$ for every j and the result follows as a consequence of the invertibility of the matrix Θ.

EXAMPLE 10.2 *(Haar-type unbalanced matrices)*
Since Haar type unbalanced matrices $H(m)$ as defined in [3] have rows with zero mean, except for the first row which is the constant row $(\frac{1}{\sqrt{p^m}}, \ldots, \frac{1}{\sqrt{p^m}})$, orthogonal matrices of the form

$$\Theta(m) = \frac{\pi}{\sqrt{p^m}} H(m)$$

satisfy Proposition 10.2. We present below two examples:

$$\Theta(3) = \frac{\pi}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$H(6) = \frac{\pi}{\sqrt{6}} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

EXAMPLE 10.3 *(Generalized Walsh-type Riesz Products)*
Since Walsh orthogonal matrices $W(2^k), k = 1, \ldots,$ produced from the Walsh system $\{w_0, \ldots, w_{2^k}\}$ defined in [6] have rows with zero mean, except for the first row which is the constant row $(1, \ldots, 1)$, orthogonal matrices of the form

$$\Theta(2^k) = \frac{\pi}{2^k} W(2^k)$$

satisfy Proposition 10.2. We present below two examples:

$$\Theta(2) = \frac{\pi}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\Theta(4) = \frac{\pi}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ldots.$$

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