Chapter 14

WALSH SERIES OF COUNTABLY MANY VARIABLES

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Abstract This paper presents some results on the convergence of the Walsh-Fourier series for the functions of countably many variables.

1. Introduction and Background Work

Let $\mathbf{T}^\infty$ and $[0,1]^\infty$ are the Cartesian product of countably many of one-dimensional tori $\mathbf{T} = \mathbf{R}/\mathbf{Z}$:

$$\mathbf{T}^\infty = \{x = (x_1, \ldots, x_n, \ldots) : 0 \le x_n < 1, n \in \mathbf{N}\}$$

and the Cartesian product of countably many segments $[0,1]$, respectively.

The torus $\mathbf{T}^\infty$ and the cube $[0,1]^\infty$ are equipped with the *Tychonoff topology*.

The Lebesque measure is defined on $\mathbf{T}^\infty$ and on $[0,1]^\infty$ as on the product of the spaces with one-dimensional Lebesque measure. Hence it is possible to consider L^p and other spaces on $\mathbf{T}^\infty$ and on $[0,1]^\infty$.

H. Steinhaus (1930) translated a probability result of A.N. Kolmogorov into terms of the theory of functions:

A trigonometric series of the form $\sum\limits_{k=1}^{\infty} a_k e^{2\pi i x_k}$ *converges (diverges) almost everywhere on* $\mathbf{T}^\infty$ *if* $\sum\limits_{k=1}^{\infty} |a_k|^2 < \infty$ *(*$\sum\limits_{k=1}^{\infty} |a_k|^2 = \infty$*).*

The system of functions

$$\theta_{n_1,\ldots,n_p}(x) = \prod_{r=1}^{p} e^{2\pi i n_r x_r}, p \in \mathbf{N}, n_r \in \mathbf{Z}, x = (x_1, \ldots, x_p, \ldots) \in \mathbf{T}^\infty$$

is called the *Jessen system*.

The Jessen system is a complete orthonormal system on $\mathbf{T}^\infty$.

B. Jessen (1934) [1] obtained the representation of an integrable function as a limit of integrals.

Let us denote by $\mathbf{T}^n$ the n-dimensional torus:

$$\mathbf{T}^n = \{(x_1, \ldots, x_n) : 0 \leq x_k < 1,\ k = 1, 2, \ldots, n\},$$

and by $\mathbf{T}^{n,\infty}$ the infinite dimensional torus:

$$\mathbf{T}^{n,\infty} = \{(x_{n+1}, \ldots, x_{n+p}, \ldots) : 0 \leq x_k < 1, \quad k = n+1, \ldots, n+p, \ldots\}.$$

Let us denote by $[0,1]^{n,\infty}$ the infinite dimensional cube

$$[0,1]^{n,\infty} = \{(x_{n+1}, \ldots, x_{n+p}, \ldots) \colon 0 \leq x_k \leq 1, \quad k = n+1, \ldots n+p, \ldots\},$$

$[0,1]^n$ — n-dimensional cube.

B. Jessen [1] proved the following

THEOREM 14.1 *Let $f \in L(\mathbf{T}^\infty)$. Then the sequence of functions*

$$f_n(x) = \int\limits_{\mathbf{T}^{n,\infty}} f(x) dx_{n+1} \ldots dx_{n+p} \ldots,\ n \in \mathbf{N}.$$

converges a.e. on $\mathbf{T}^\infty$ to f, that is

$$\lim_{n\to\infty} \int\limits_{\mathbf{T}^{n,\infty}} f(x) dx_{n+1} \ldots dx_{n+p} \ldots = f(x), \quad a.e.\ x \in \mathbf{T}^\infty.$$

If $f \in L^p(\mathbf{T}^\infty), p \geq 1$, then $f_n \in L^p(\mathbf{T}^\infty)$ and sequence $\{f_n\}$ converges to f in $L^p(\mathbf{T}^\infty)$.

Of course, Jessen's theorem is correct for functions $f \in L([0,1]^\infty)$ and

$$f_n(x) = \int_{[0,1]^{n,\infty}} f(x)\, dx_{n+1} \ldots dx_{n+p} \ldots, \quad n \in \mathbf{N}. \tag{14.1}$$

This Jessen's theorem allows to solve some questions on Fourier series of functions on $\mathbf{T}^\infty$ and $[0,1]^\infty$.

Let us remark that T.I. Ahobadze (1986) considered systems of functions on $\mathbf{T}^\infty$ and in particular the Jessen system.

2. Series in Terms of Infinite Dimensional Walsh System

We will consider Walsh system on $[0,1]^\infty$:

$$W_{n_1,\ldots,n_p}(x) = \prod_{r=1}^{p} w_{n_r}(x_r), p \in \mathbf{N}, n_r \in \mathbf{Z}_+, x = (x_1, \ldots, x_p, \ldots) \in [0,1]^\infty,$$

where $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$ and $w_{n_r}(x_r)$ are one-dimensional Walsh functions.

This system is a complete orthonormal system on $[0,1]^\infty$.

F. Schipp [2] considered generalized product systems and obtained results on L^p - norm convergence $(1 < p < \infty)$ of Fourier expansions with respect to the product system of independent systems and with respect to Vilenkin systems.

Let ${\mathbf{Z}_+}^{<\infty}$ be the set of infinite dimensional vectors $n = (n_1, \ldots, n_p, \ldots)$ with $n_p \in \mathbf{Z}_+$ and only a finite number of n_p are different from zero.

We will consider series on infinite dimensional Walsh system

$$\sum_{n \in {\mathbf{Z}_+}^{<\infty}} a_n W_n(x), \qquad \text{where } x \in [0,1]^\infty,\ a_n \in \mathbf{R}. \tag{14.2}$$

If for $n = (n_1, \ldots, n_p, \ldots) \in {\mathbf{Z}_+}^{<\infty}$ the coordinates $n_k = 0$ for $k > p$, then we'll use the following notations for a_n: $a_n = a_{n_1,\ldots,n_p}$. Each of these notation determines n and a_n uniquely.

Denote the rectangular partial sums by

$$S_{p,N_1,\ldots,N_p}(x) = \sum_{n_1=0}^{N_1} \ldots \sum_{n_p=0}^{N_p} a_{n_1,\ldots,n_p} w_{n_1}(x_1) \ldots w_{n_p}(x_p), \tag{14.3}$$

where $p \in \mathbf{N}, N_1, \ldots, N_p \in \mathbf{Z}_+$, $x \in [0,1]^\infty$. If $N_1 = \ldots = N_p$ these are cubic partial sums.

We say that the series (14.2) converges rectangularly at the point $x \in [0,1]^\infty$ to the number s if for any $\varepsilon > 0$ there exists an index P such that for any $p \geq P$ we can find a natural number N such that for any $N_1, \ldots, N_p \geq N$ the inequality $|S_{p,N_1,\ldots,N_p}(x) - s| < \varepsilon$ is fulfilled.

The series (14.2) is called convergent over cubes if we consider only cubic partial sums in this definition.

We say that the series (14.2) converges rectangularly (over cubes) in strengthened sense at the point $x \in [0,1]^\infty$ to the number s if for any p multiple Walsh series $\sum\limits_{n_1,\ldots n_p} a_{n_1,\ldots,n_p} w_{n_1}(x_1) \ldots w_{n_p}(x_p)$ converges rectangularly (over cubes) at the point x and there exist limit

$$\lim_{p \to \infty} \sum_{n_1,\ldots,n_p} a_{n_1,\ldots,n_p} w_{n_1}(x_1) \ldots w_{n_p}(x_p) = s.$$

It is easy to see if the series (14.2) converges in the strengthened sense, then it converges.

F. Móricz [3] proved that if $f \in L^2([0,1]^2)$, then the square partial sums of the Walsh-Fourier series of f converge to f almost everywhere on $[0,1]^2$. This result is correctly and in d-dimensional case. It follows from this and from Jessen's theorem A

THEOREM 14.2 *If $f \in L^2([0,1]^\infty)$, then the Fourier series of f converges over cubes in strengthened sense to f a.e. on $[0,1]^\infty$.*

Let $\varphi\colon [0,+\infty) \to [0,+\infty)$ be an increasing function. Then we denote by $\varphi(L)([0,1]^\infty)$ the set of all measurable functions f on $[0,1]^\infty$ such that $\int\limits_{[0,1]^\infty} \varphi(|f(x)|)dx < \infty$. Similarly, $\varphi(L)([0,1]^n)$ is the set of all measurable functions f on $[0,1]^n$, such that $\int\limits_{[0,1]^n} \varphi(|f(x)|)\, dx_1 \ldots dx_n < \infty$.

LEMMA 14.1 *Let $\varphi\colon [0,+\infty) \to [0,+\infty)$ be an increasing convex function. If $f \in \varphi(L)([0,1]^\infty)$, then f_n ($n \in \mathbf{N}$) (see (14.1)) are from $\varphi(L)([0,1]^n)$. If we consider f_n as functions on $[0,1]^\infty$, then $f_n \in \varphi(L)([0,1]^\infty)$, $n \in \mathbf{N}$.*

Proof. Recall the Jensen inequality in general form. Let μ be probability measure on measurable space $(X, \mathcal{A})$, g — μ-integrable function with values in the domain of definition of convex function ψ and $\psi(f)$ is integrable function. Then,

$$\psi(\int_X g(x)\mu(dx)) \leq \int_X \psi(f(x))\mu(dx).$$

We have in our case $X = [0,1]^\infty$, μ — Lebesque measure, $g = |f_n|$, $\varphi = \psi$. It follows from increasing of function φ and from Jensen's inequality that

$$\begin{aligned} \varphi(|f_n(x)|) &\leq \varphi(\int_{[0,1]^{n,\infty}} |f(x)|\, dx_{n+1} \ldots dx_{n+p} \ldots) \\ &\leq \int_{[0,1]^{n,\infty}} \varphi(|f(x)|)\, dx_{n+1} \ldots dx_{n+p} \ldots . \end{aligned}$$

Now we obtain by integrating this inequality on $[0,1]^n$

$$\int_{[0,1]^n} \varphi(|f_n(x)|)\, dx_1 \ldots dx_n \leq \int_{[0,1]^\infty} \varphi(|f(x)|)\, dx < \infty.$$

Lemma is proven.

It is known and follows from the Jessen-Marcinkiewich-Zygmund theorem on the strong differentiation of integrals [4], Ch. XVII, paragraph 2, and from the form of partial sums of multiple Fourier-Haar series that if $f \in L(\log^+ L)^{d-1}([0,1]^d)$, then rectangular partial sums of the Haar-Fourier series of f converge to f almost everywhere on $[0,1]^d$.

For Walsh-Fourier series we obtain by this and by Lemma and Theorem A the following

THEOREM 14.3 *If $f \in \bigcap_{d=1}^{\infty} L(\log^+ L)^d(\mathbf{T}^\infty)$, then the rectangular partial sums $S_{2^{n_1},2^{n_2},\ldots,2^{n_p}}$ of the Walsh-Fourier series of f converges in strengthened sense to f a.e. on $[0,1]^\infty$.*

REMARK 14.1 *The condition $f \in \bigcap_{d=1}^{\infty} L(\log^+ L)^d([0,1]^\infty)$ may be given in the equivalent form: there exist an increasing function $h(x)$ defined on $[0,\infty)$, $h(x) \geq 0$, $\lim\limits_{x\to\infty} h(x) = +\infty$ such that*

$$\int\limits_{[0,1]^\infty} |f(x)|(\log^+ |f(x)|)^{h(|f(x)|)} dx < \infty.$$

REMARK 14.2 *It follows from S. Saks [4], Ch. XVII, paragraph 2, and T.S. Zerekidze [5] results that any class $L(\log^+ L)^d([0,1]^\infty)$ contains a function for which the Theorem 14.3 is not correct.*

REMARK 14.3 *It follows from results by D.K. Sanadze and Sh.V. Kheladze [6] that under conditions of Theorem 14.3 the rectangular partial sums of the Walsh-Fourier series of f converge to f a.e. on $[0,1]^\infty$ if all but one indices are lacunary.*

Let us compare the Theorem 14.3 with the result for the Jessen system

THEOREM 14.4 *If $f \in \bigcap_{d=1}^{\infty} L(\log^+ L)^d(\mathbf{T}^\infty)$, then the trigonometric Fourier series of f converges over cubes in the strengthened sense to f a.e. on $\mathbf{T}^\infty$.*

(The proof of Theorem 14.4 can be found in [7].)

REMARK 14.4 *It follows from Konyagin's result [8] that any class $L(\log^+ L)^d(\mathbf{T}^\infty)$ contains a function with the Fourier series divergent over cubes everywhere on $\mathbf{T}^\infty$.*

F. Weisz proved (see in [9] the 2-dimensional case) that if $f \in L[0,1]^n$ and

$$S_{k,\ldots,k}(f,x) = \sum_{m_1=0}^{k-1} \cdots \sum_{m_1=0}^{k-1} a_{m_1,\ldots,m_n} w_{m_1}(x_1)\ldots w_{m_n}(x_n)$$

are cubic partial sums of Fourier series of f ($k = 1,2,3,\ldots$), then

$$\lim_{k\to\infty} \frac{S_{1,\ldots,1}(f,x) + S_{2,\ldots,2}(f,x) + \ldots + S_{k,\ldots,k}(f,x)}{k} = f(x) \quad \text{a.e. on } [0,1]^n.$$

From this and from Theorem 14.1 we obtain:

If $f \in L([0,1]^\infty)$ and $S_{n,k,\ldots,k}(f,x)$ cubic partial sums of f (see (3)) then

$$\lim_{n\to\infty} \lim_{k\to\infty} \frac{S_{n,1,\ldots,1}(f,x) + S_{n,2,\ldots,2}(f,x) + \ldots + S_{n,k,\ldots,k}(f,x)}{k} = f(x)$$

a.e. on $[0,1]^\infty$.

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