Chapter 4

GENERALIZED INTEGRALS IN WALSH ANALYSIS

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Abstract Generalized integrals with respect to multidimensional dyadic basis are considered and applied to recover coefficients of multiple series in Haar and Walsh systems on $[0,1]^m$ and on group G^m

1. Introduction

In this paper we survey some results related to the problem of recovering the coefficients of multiple Walsh and Haar series. Generalized integrals which solve this problem are defined in terms of the dyadic derivation basis.

It is known that similarly to the case of series in trigonometrical system (see [17]), Walsh and Haar series being convergent everywhere can fail to be the Fourier-Lebesgue series of their sums. Therefore the coefficients problem requires integration processes more general than the Lebesgue one.

A history of this theory, especially in the one-dimensional case, was presented in [14].

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We concentrate here on the multidimensional case. In this case a solution of the coefficients problem essentially depends on the type of convergence of multiple series, on the set of convergence and also on the domain on which Walsh system is defined (on the dyadic group or on the unit interval in $\mathbf{R}^m$).

A version of multidimensional Perron integral solving the coefficients problem for rectangular convergent multiple Haar and Walsh series in $\mathbf{R}^m$ was considered in [13]. We discuss this case of convergence in more details in Section 5, both in group and in $\mathbf{R}^m$ setting. The coefficient problem in the case of a more general ρ-regular rectangular convergence was considered in several paper by the first author. These results are surveyed in Section 6.

2. Henstock- and Perron-type Integrals with Respect to a Derivation Basis

We remind the principal elements of the Henstock theory of integration (see [3]).

A *derivation basis* (or simply a *basis*) $\mathcal{B}$ in a measure space $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of measurable subsets of X having positive measure μ and called *generalized intervals* or *$\mathcal{B}$-intervals*. That is, $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ so that each $\beta \in \mathcal{B}$ is a set of pairs (I, x), where $I \in \mathcal{I}$, $x \in X$, and $\mathcal{B}$ has the *filter base property*: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So each basis is a directed set with the order given by "reversed" inclusion. We shall refer to the elements β of $\mathcal{B}$ as *basis sets*. We suppose that $x \in I$ for all the pairs (I, x) constituting each $\beta \in \mathcal{B}$. For a set $E \subset X$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) := \{(I, x) \in \beta : I \subset E\} \text{ and } \beta[E] := \{(I, x) \in \beta : x \in E\}.$$

Certain additional hypotheses guarantee some nice properties of a basis. For example, it is useful to suppose that the basis $\mathcal{B}$ *ignores no point*, i.e., $\beta[\{x\}] \neq \emptyset$ for any point $x \in X$ and for any $\beta \in \mathcal{B}$.

If X is a metric or a topological space it is supposed that $\mathcal{B}$ is a *Vitali basis* by which we mean that for any x and for any neighborhood $U(x)$ of x there exists $\beta_x \in \mathcal{B}$ such that $I \subset U(x)$ for each pair $(I, x) \in \beta_x$.

A *β-partition* is a finite collection π of elements of β, where the distinct elements (I', x') and (I'', x'') in π have I' and I'' disjoint (or at least non-overlapping, i.e., $\mu(I' \cap I'') = 0$). Let $L \in \mathcal{I}$. If $\pi \subset \beta(L)$ then π is called β-partition *in L,* if $\bigcup_{(I,x)\in\pi} I = L$ then π is called β-partition *of L.*

We say that a basis $\mathcal{B}$ has the *partitioning property* if the following conditions hold: (i) for each finite collection $I_0, I_1, ..., I_n$ of $\mathcal{B}$-intervals with $I_1, ...I_n \subset I_0$ the difference $I_0 \setminus \bigcup_{i=1}^{n} I_i$ can be expressed as a finite union of pairwise non-overlapping $\mathcal{B}$-intervals; (ii) for each $\mathcal{B}$-interval I and for any $\beta \in \mathcal{B}$ there exists a β-partition of I.

DEFINITION 4.1 Let $\mathcal{B}$ be a basis having the partitioning property and $L \in \mathcal{I}$. A function f on L is said to be *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if for every $\varepsilon > 0$, there exists $\beta \in \mathcal{B}$ such that for any β-partition π of L we have:

$$\left| \sum_{(I,x)\in\pi} f(x)\mu(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

It is easy to check that if a function f is $H_{\mathcal{B}}$-integrable on L, then it is also integrable on each $\mathcal{B}$-subintervals of L and so the indefinite $H_{\mathcal{B}}$-integral is defined as an additive $\mathcal{B}$-interval function.

The following extension of the previous definition is useful in many cases.

DEFINITION 4.2 A function f defined almost everywhere on $L \in \mathcal{I}$ is *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if the function

$$f_1(x) := \begin{cases} f(x), & \text{if it is defined,} \\ 0, & \text{otherwise,} \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and its $H_{\mathcal{B}}$-integral is equal A.

Let F be an additive set function on $\mathcal{I}$ and E an arbitrary subset of X. For a fixed $\beta \in \mathcal{B}$, we set

$$Var(E, F, \beta) := \sup_{\pi \subset \beta[E]} \sum |F(I)|.$$

We put also

$$V_F(E) = V(E, F, \mathcal{B}) := \inf_{\beta\in\mathcal{B}} Var(E, F, \beta).$$

The extended real-valued set function $V_F(\cdot)$ is called *variational measure* generated by F, with respect to the basis $\mathcal{B}$. It is an outer measure and, in the case of a metric space X, a metric outer measure (in the last case it should be assumed that the basis is a Vitali basis).

Given a set function $F : \mathcal{I} \to \mathbf{R}$ we define the *upper* and *lower $\mathcal{B}$-derivative* at a point x, with respect to the basis $\mathcal{B}$ and measure μ, as

$$\overline{D}_{\mathcal{B}}F(x) := \inf_{\beta\in\mathcal{B}} \sup_{(I,x)\in\beta} \frac{F(I)}{\mu(I)} \quad \text{and} \quad \underline{D}_{\mathcal{B}}F(x) := \sup_{\beta\in\mathcal{B}} \inf_{(I,x)\in\beta} \frac{F(I)}{\mu(I)}, \tag{4.1}$$

respectively. As we have assumed that $\mathcal{B}$ ignores no point then it is always true that $\overline{D}_{\mathcal{B}}F(x) \geq \underline{D}_{\mathcal{B}}F(x)$. If $\overline{D}_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x)$ we call this common value *$\mathcal{B}$-derivative* $D_{\mathcal{B}}F(x)$.

We say that a set function F is *$\mathcal{B}$-continuous* at a point x, with respect to the basis $\mathcal{B}$, if $V_F(\{x\}) = 0$.

We shall need the following (see [3, Proposition 1.6.4])

PROPOSITION 4.1 *Let an additive function $F : \mathcal{I} \to \mathbf{R}$ be $\mathcal{B}$-differentiable on $L \in \mathcal{I}$ outside a set $E \subset L$ such that $V_F(E) = 0$. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x), & \text{if it exists}, \\ 0, & \text{if } x \in E, \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

The next theorem is a corollary of the above proposition.

THEOREM 4.1 *Let an additive function $F : \mathcal{I} \to \mathbf{R}$ be $\mathcal{B}$-differentiable everywhere on $L \in \mathcal{I}$ outside of a set E with $\mu(E) = 0$, and $-\infty < \underline{D}_{\mathcal{B}}F(x) < \overline{D}_{\mathcal{B}}F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}$-continuous. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x), & \text{if it exists}, \\ 0, & \text{if } x \in E, \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

To define a Perron-type integral with respect to a basis $\mathcal{B}$ we remind that a $\mathcal{B}$-interval function F is called *$\mathcal{B}$-superadditive* (resp. *$\mathcal{B}$-subadditive*) if every finite collection $\{I_i\}_{i=1}^{p}$ of pair-wise non-overlapping $\mathcal{B}$-intervals such that $\bigcup_{i=1}^{p} I_i \in \mathcal{I}$ satisfies

$$\sum_{i=1}^{p} F(I_i) \le F\left(\bigcup_{i=1}^{p} I_i\right) \ \text{(resp.}\ \sum_{i=1}^{p} F(I_i) \ge F\left(\bigcup_{i=1}^{p} I_i\right)\ ).$$

By $\overline{\mathcal{A}}_{\mathcal{B}}$ (resp. $\underline{\mathcal{A}}_{\mathcal{B}}$) denote the set of all $\mathcal{B}$-superadditive (resp. $\mathcal{B}$-subadditive) functions. A $\mathcal{B}$-interval function F is called *$\mathcal{B}$-additive* if $F \in \overline{\mathcal{A}}_{\mathcal{B}} \cap \underline{\mathcal{A}}_{\mathcal{B}}$. Let $\mathcal{A}_{\mathcal{B}}$ denote the set of all $\mathcal{B}$-additive functions.

With this notation we introduce the following definition of a Perron-type integral.

DEFINITION 4.3 A function f defined on $L \in \mathcal{I}$ is said to be *$P_{\mathcal{B}}$-integrable* on L if for every $\varepsilon > 0$ there exist $\mathcal{B}$-interval functions $M \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $m \in \underline{\mathcal{A}}_{\mathcal{B}}$ such that

$$\underline{D}_{\mathcal{B}}M(x) \ge f(x) \ge \overline{D}_{\mathcal{B}}m(x) \ \text{ for each } \ x \in L \tag{4.2}$$

and $M(L) - m(L) < \varepsilon$. The value of the *$P_{\mathcal{B}}$-integral* on L is $(P_{\mathcal{B}}) \int_L f := \inf_M M(L) = \sup_m m(L)$.

This integral is known (see [3]) to be equivalent to the $H_{\mathcal{B}}$-integral. If we want the inequality 4.2 in this definition to hold not everywhere but with some exceptional set then we need to assume some kind of continuity of the functions M and m on this set. We consider below several generalizations of the $P_{\mathcal{B}}$-integral in this direction.

3. Dyadic Derivation Bases in $[0,1]^m$ and in Group G^m

We consider here derivation bases in two spaces, according to two types of domains on which the Walsh system can be defined. The first one is the unit cube $[0,1]^m$ in which we consider the dyadic basis. Another example will be a basis in the dyadic group G or in its cartesian product G^m.

In the case of $X = [0,1]$ the family $\mathcal{I}$ of $\mathcal{B}$-intervals is constituted by dyadic intervals

$$J_j^{(n)} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \;\; 0 \le j \le 2^n - 1, \;\; n = 0, 1, 2, \dots.$$

Here n is a *rank* of the interval.

If $X = [0,1]^m$, $\mathcal{B}$-intervals are defined as m-dimensional dyadic intervals

$$J_{\mathbf{j}}^{(\mathbf{n})} := J_{j_1}^{(n_1)} \times \ldots \times J_{j_m}^{(n_m)} \tag{4.3}$$

where $\mathbf{j} = (j_1, \dots, j_m)$ and $\mathbf{n} = (n_1, \dots, n_m)$, with $\mathbf{n}$ being a rank of the interval. We denote the family of all these intervals by $\mathcal{I}_d$.

To define a dyadic basis it is enough to define basis sets β. For $X = [0,1]$ we put

$$\beta_\delta := \{I \in \mathcal{I}_d : \;\; I \subset U(x, \delta(x))\},$$

where δ is a so-called gauge, i.e., a positive function defined on X, and $U(x, \delta)$ denotes the neighborhood of x of radius δ. So the *dyadic basis* is defined as $\mathcal{B}_d := \{\beta_\delta : \;\; \delta : X \to (0, \infty)\}$.

In the m-dimensional case we consider two dyadic basis. The first one is defined exactly as above with $\mathcal{I}_d$ being the family of all m-dimensional dyadic intervals. The second one is called a regular dyadic basis. To define it we use the notion of regularity. The *parameter of regularity of a dyadic interval* of the form (4.3) is defined as

$$\min_{i,l}\{|J_{j_i}^{(n_i)}|/|J_{j_l}^{(n_l)}|\}.$$

Analogously the *parameter of regularity of a vector* $\mathbf{a} = (a_1, \dots, a_m)$ is defined as

$$\min_{i,l}\{a_i/a_l\}.$$

We write $reg(J)$ (resp. $reg(\mathbf{a})$) for the parameter of regularity of a dyadic interval J (resp. of a vector $\mathbf{a}$).

Now basis sets of *ρ-regular dyadic basis $\mathcal{B}_{d,\rho}$* we define as

$$\beta_{\delta,\rho} := \{I \in \mathcal{I}_d : \ I \subset U(x, \delta(x)), \ reg(I) \geq \rho\} .$$

Applying Definition 4.1 to these dyadic bases we obtain *$H_{\mathcal{B}_d}$-integral* (the *dyadic Henstock integral*) and *$H_{\mathcal{B}_{d,\rho}}$-integral* (the *ρ-regular dyadic Henstock integral*).

Now we turn to a group setting.

Recall (see [1, 2, 11]) that the *dyadic group* G is a set of sequences $t = \{t_i\}_{i=0}^{\infty}$ where $t_i = 0$ or 1 with group operation in G being defined as the coordinate-wise addition $(mod\,2)$. The topology in G is defined by a chain of subgroups $G_k = \{t = \{t_i\} : t_i = 0, i \leq k\}, \ \ k = 0, 1, \ldots$, so that $G = G_0$ and $\{0\} = \bigcap_{n=0}^{\infty} G_n$. With respect to this topology, the subgroups G_n are clopen sets and G is a zero-dimensional compact abelian group. The factor group G/G_n contains 2^n elements. We denote by K_n any coset of the subgroup G_n and by $K_n(a)$ the coset of the subgroup G_n which contains an element $a = \{a_i\}_{i=0}^{\infty}$, i.e., $K_n(a) := a + G_n = \{t = \{t_i\} : t_i = a_i, i \leq k\}$. In particular $G_n = K_n(0)$. For each $a \in G$ the sequence $\{K_n(a)\}$ is decreasing and $\{a\} = \bigcap_n K_n(a)$.

In the product space G^m we consider, similarly to the case of the m-dimensional cube, two types of $\mathcal{B}$-intervals. By $\mathcal{I}_{G^m}$ we denote a family of all the sets of the form

$$K_{\mathbf{n}} := K_{n_1} \times \ldots \times K_{n_m}$$

where $\mathbf{n} = (\mathbf{n_1}, \ldots, \mathbf{n_m})$ is a *rank* of this $\mathcal{B}$-interval. If $\mathbf{t} \in G^m$ then

$$K_{\mathbf{n}}(\mathbf{t}) := K_{n_1}(t_1) \times \ldots \times K_{n_m}(t_m)$$

If we assume here that $reg(\mathbf{n}) \geq \rho$ for some $\rho \in (0, 1]$, then we obtain the family $\mathcal{I}_{G^m}^{\rho}$ of ρ-regular $\mathcal{B}$-intervals. Accordingly we get two derivation bases in G^m. A basis $\mathcal{B}_{G^m}$ is constituted by basis sets

$$\beta_\nu := \{(I, \mathbf{t}) : \mathbf{t} \in G^m, \ \ I = K_{\mathbf{n}}(\mathbf{t}), \ \ \min n_i \geq \nu(\mathbf{t})\}$$

where ν runs over the set of all integer-valued functions $\nu : G^m \to \mathbf{N}$ and $\mathbf{n} = (n_1, \ldots, n_m)$. A ρ-regular basis $\mathcal{B}_{G^m}^{\rho}$ is constituted by basis sets

$$\beta_\nu^\rho := \{(I, \mathbf{t}) \in \beta_\nu : I \in \mathcal{I}_{G^m}^{\rho}\}.$$

These two bases have all the properties of a general derivation basis. The partitioning property follows easily from compactness of any $\mathcal{B}_{G^m}$-interval by standard methods (see [3]).

Using the normalized Haar measures on the group G we denote by μ the product measure on G^m. Then $\mu(K_{\mathbf{n}}) = 2^{-(n_1+\ldots+n_m)}$ where $\mathbf{n} = (n_1, \ldots, n_m)$.

Definition 4.1 of the $H_{\mathcal{B}}$-integral can be rewritten for the bases in G^m in the following form (see [15]):

DEFINITION 4.4 Let $L \in \mathcal{I}_{G^m}$. A function f defined on L is said to be H_{G^m}*-integrable* (resp. $H^{\rho}_{G^m}$*-integrable*) on L, with integral value A, if for every $\varepsilon > 0$, there exists a function $\nu : L \to \mathbf{N}$ such that for any β_ν-partition (resp. β^{ρ}_{ν}-partition) π of L we have:

$$\left| \sum_{(I,\mathbf{t})\in\pi} f(\mathbf{t})\mu(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{G^m}) \int_L f$
(resp. by $(H^{\rho}_{G^m}) \int_L f$).

Note that in the case of our basis $\mathcal{B}_G$, given a point $\mathbf{t}$, any β_ν-partition contains only one pair $(I, \mathbf{t})$ with this point $\mathbf{t}$. Because of this we can reformulate the definition of $\mathcal{B}$-continuity in a simpler way, saying that a set function F is $\mathcal{B}_G$*-continuous* at a point $\mathbf{t}$, with respect to the basis $\mathcal{B}_G$, if $\lim_{n\to\infty} F(K_n(\mathbf{t})) = \mathbf{0}$.

The map

$$\Phi : t \mapsto x = \sum_{j=1}^{\infty} \frac{t_i}{2^{i+1}} \tag{4.4}$$

is one-to-one correspondence between the group G and the interval $[0,1]$, up to a countable set. Indeed, denoting by Q_d the set of all dyadic rational points in $[0,1]$, i.e., points of the form $\frac{j}{2^k}$, $0 \le j \le 2^k$, $k = 0, 1, \ldots$, we note that each $x \in Q_d$ has two expansions, a finite one and an infinite one. If we exclude from G the elements corresponding to one type of expansion, for example to the infinite one, then the correspondence (4.4) is one-to-one and the converse mapping Φ^{-1} is defined on [0,1). The function Φ maps each $\mathcal{B}_{G^m}$-interval K_n onto a dyadic interval $J^{(n)}_j$. So there is a closed relation between bases $\mathcal{B}_d$ and $\mathcal{B}_{G^m}$. But as we shall see below, the fact that G^m is a zero-dimensional space while $[0,1]^m$ is connected, implies an essential difference in the properties of the integrals defined with respect to those bases. The principal difference can be seen already in the one-dimensional case. In the case of G, we can associate with each point $t \in G$, a unique sequence of nested $\mathcal{B}_G$-intervals $K_n(t)$ converging to t. We call it the *basic sequence convergent to* t. But in the case of $X = [0,1]$ such a unique sequence of $\mathcal{B}_d$-intervals can be associated with a point x only in the case x is dyadic-irrational. If $x \in Q_d$, then we can associate with it two basic sequences of dyadic intervals: the left one and the right one for which x is the common end-point, starting with some rank n.

In the product space G^m or $[0,1]^m$ the m-multiple sequence $\{I_{\mathbf{n}}\}$ of $\mathcal{B}$-intervals is a *basic sequence convergent to* $\mathbf{t} \in G^m$ (resp. *to* $\mathbf{x} \in [0,1]^m$) if $I_{\mathbf{n}} = I_{n_1} \times \ldots \times I_{n_m}$ with $\{I_{n_i}\}$ being the one-dimensional basic sequence convergent to $t_i \in G$ (resp. to $x_i \in [0,1]$). Accordingly in G^m we have only one basic sequence convergent to each $\mathbf{t}$ while in $[0,1]^m$ the number of basic sequences convergent to $\mathbf{x}$ is equal 2^s, $0 \le s \le m$, if $\mathbf{x}$ has s dyadic-rational coordinates.

4. Multiple Walsh and Haar Series

The *Walsh functions* (in Paley numeration) on G (see [2, 11]) are defined by

$$w_n(t) := (-1)^{\sum\limits_{i=0}^{\infty} t_i \varepsilon_i^{(n)}}$$

where

$$t = \{t_i\} \in G, \qquad n = \sum_{i=0}^{\infty} 2^i \varepsilon_i^{(n)} \ \ (\varepsilon_i^{(n)} \in \{0,1\}).$$

Using mapping Φ^{-1} considered above, we can define Walsh system on the unit interval as $w(\Phi^{-1}(x))$. For these functions we shall use the same notation: $w(x)$.

The *Haar functions* are usually considered on $[0,1)$. But in case of need we can always pass to the group setting using the same mapping Φ. We put $\chi_0(x) \equiv 1$. If $n = 2^k + i$ ($k = 0, 1, \ldots$, $i = 0, \ldots, 2^k - 1$), we put

$$\chi_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in \left[\frac{2i}{2^{k+1}}, \frac{2i+1}{2^{k+1}}\right), \\ -2^{k/2}, & \text{if } x \in \left[\frac{2i+1}{2^{k+1}}, \frac{2i+2}{2^{k+1}}\right), \\ 0, & \text{if } x \in [0,1) \setminus \left[\frac{2i}{2^{k+1}}, \frac{2i+2}{2^{k+1}}\right). \end{cases}$$

An *m-dimensional Walsh* (resp. *Haar*) *series* (both on G^m and on $[0,1]^m$) is defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \ldots \sum_{n_m=0}^{\infty} b_{n_1,\ldots,n_m} \prod_{i=1}^{m} w_{n_i}(x_i) \tag{4.5}$$

$$\left(\text{resp. } \sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \ldots \sum_{n_m=0}^{\infty} a_{n_1,\ldots,n_m} \prod_{i=1}^{m} \chi_{n_i}(x_i) \ \right) \tag{4.6}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real numbers. If $\mathbf{N} = (N_1, \ldots, N_m)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of series (4.5) (resp. (4.6)) at a point $\mathbf{x} = (x_1, \ldots, x_m)$

is

$$S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} b_{\mathbf{n}} \omega_{\mathbf{n}}(\mathbf{x}) \quad (\text{resp. } S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} a_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x})\ ).$$

The series (4.5) (or (4.6)) *rectangularly converges* to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \text{ as } \min_i \{N_i\} \to \infty.$$

We consider also the regular convergence of series. Let $\rho \in (0, 1]$; then the series (4.5) (or (4.6)) *ρ-regularly converges* to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \text{ as } \min_i \{N_i\} \to \infty \text{ and } reg(\mathbf{N}) \geq \rho.$$

It is obvious that if the series (4.5) (or (4.6)) rectangularly converges to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ then for every $\rho \in (0, 1]$ this series ρ-regularly converges to $S(\mathbf{x})$ at $\mathbf{x}$.

A starting point for an application of the dyadic derivative and the dyadic integral to the theory of Walsh and Haar series is an observation that due to martingale properties of the partial sums $S_{2^{\mathbf{k}}}$ of those series (here $2^{\mathbf{k}}$ stand for $(2^{k_1}, \dots, 2^{k_m})$) the integral $\int_{I^{(\mathbf{k})}} S_{2^{\mathbf{k}}}$ where $I^{(\mathbf{k})}$ is a $\mathcal{B}$-interval of rank k either in $[0, 1]^m$ or in G^m, defines an additive $\mathcal{B}$-interval function $\psi(I)$ on the family $\mathcal{I}$ of all $\mathcal{B}$ intervals. (In dyadic analysis the function ψ is sometimes referred to as *quasi-measure* (see [11, 16]).) Since the sum $S_{2^{\mathbf{k}}}$ is constant on each $I^{(\mathbf{k})}$ (in the interior of $I^{(\mathbf{k})}$ in the case of $[0, 1]^m$) we get

$$S_{2^{\mathbf{k}}}(\mathbf{x}) = \frac{1}{|I^{(\mathbf{k})}|} \int_{I^{(\mathbf{k})}} S_{2^{\mathbf{k}}} = \frac{\psi(I^{(\mathbf{k})})}{|I^{(\mathbf{k})}|} \tag{4.7}$$

for any point $\mathbf{x} \in I^{(\mathbf{k})}$.

Another simple observation which is essential for proving that a given Walsh or Haar series is the Fourier series in the sense of some general integral, is the following statement (see [14, Proposition 4]).

PROPOSITION 4.2 *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}_d$ or $\mathcal{I}_{G^m}$. Assume a series of the form* (4.5) *or* (4.6) *is given. Let the $\mathcal{B}$-interval function ψ be defined for this series by* (4.7)*. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_{\mathcal{I}} \{$ for any $\mathcal{B}$-interval I.*

It is seen from formula (4.7) that for any point $\mathbf{x}$ in G^m or at least for points with dyadic-irrational coordinates, in the case of $[0, 1]^m$, rectangular (respectively, ρ-regular rectangular) convergence of the series 4.5 (or (4.6)) at a point $\mathbf{x}$ to a sum $f(\mathbf{x})$ implies $\mathcal{B}$-differentiability (respectively, $\mathcal{B}_\rho$-differentiability) of the function ψ in $\mathbf{x}$ with $f(\mathbf{x})$ being the value of the $\mathcal{B}$-derivative (resp. $\mathcal{B}_\rho$-derivative).

So in order to solve the coefficient problem it is enough to show that the function ψ is an integral of its derivative which exists at least almost everywhere. Then in view of Proposition 4.2 we get

THEOREM 4.2 *If the series* (4.5) (*or* (4.6)) *is rectangular* (*respectively, ρ-regular rectangular*) *convergent to a sum f almost everywhere on* $[0,1])^m$ *or on G^m, outside a set E such that $V_\psi(E) = 0$, then the function f is $H_\mathcal{B}$-integrable* (*respectively, $H_{\mathcal{B}_\rho}$-integrable and* (4.5) (*or* (4.6)) *is the Fourier series of f, in the sense of the respective integral.*

To use this theorem we need some additional information related to the behavior of a series on the exceptional set which would imply that the variational measure V_ψ is equal zero on this set. Such a nice behavior of ψ on the exceptional set can be obtained either from a convergence condition or from some additional growth assumptions imposed on the series. For example, it can be easily shown, in the one-dimensional case, that if the coefficients of a series 4.5 satisfy the condition $\lim_{n\to\infty} b_n = 0$ (which is a consequence of the convergence of the series at least at one dyadic-irrational point) then ψ is $\mathcal{B}_d$-continuous everywhere on $[0,1]$, and we apply Theorem 4.1 to get

THEOREM 4.3 *If the series* (4.5) (*in one dimension*) *is convergent to a sum f at each dyadic irrational point of* $[0,1]$, *then f is $H_{\mathcal{B}_d}$-integrable and* 4.5 *is the $H_{\mathcal{B}_d}$-Fourier series of f, i.e.,*

$$b_n = (H_{\mathcal{B}_d}) \int_{[0,1]} f w_n.$$

5. Coefficients Problem for Rectangular Convergent Series

In the case of the group setting, the equality (4.7) establishes the equivalence of rectangular convergence of the series (4.5) and (4.6) with respect to subsequence $2^{\mathbf{k}}$ and $\mathcal{B}_G$-differentiability of the associated function ψ *at each point of G*. So the problem of recovering the coefficients of everywhere convergent series is reduced in this case to the problem of recovering the primitive from the $\mathcal{B}_G$-derivative. So in this case we have

THEOREM 4.4 *If the series* (4.5) (*or* (4.6)) *is rectangular convergent to a sum f everywhere on G^m, then the function f is H_G-integrable and* (4.5) (*or* (4.6)) *is the Fourier series of f, in the sense of the H_G-integral.*

If we consider the series (4.5) and (4.6) on $[0,1]^m$ then the rectangular convergence everywhere does not guarantee the differentiability of the function ψ

everywhere. This function can fail to be differentiable on the set of points having at least one dyadic-rational coordinate. This exceptional set is not countable. So we can not apply Theorem 4.1 to get a multidimensional generalization of Theorem 4.3. Moreover it can be shown that $\mathcal{B}_d$-continuity of ψ on the exceptional set which follows from the convergence of the series is not enough to solve the problem of recovering the primitive.

But in this case a stronger type of continuity can be proved, namely the continuity in the sense of Saks.

DEFINITION 4.5 A $\mathcal{B}_d$-interval function ψ is called *continuous in the sense of Saks* if $\lim \psi(I) \to 0$ as $|I| \to 0$.

The next statement follows from [12].

PROPOSITION 4.3 *Suppose the series 4.5 everywhere rectangularly converges to a finite sum. Then the function ψ constructed for this series by 4.7 is continuous in the sense of Saks.*

Unfortunately continuity in the sense of Saks at the points of an exceptional set is not enough to recover the primitive by $H_{\mathcal{B}_d}$-integral. A reason for this is a fact that continuity of a function ψ in the sense of Saks on a set of measure zero does not imply that the variational measure V_ψ of this set is equal zero. So we can not use Theorem 4.4. Nevertheless the problem can be solved by a Perron-type integral which is a generalization of $P_{\mathcal{B}_d}$-integral.

DEFINITION 4.6 A function f defined on $[0,1]^m$ is said to be $\overline{P}_{\mathcal{B}_d}$*-integrable* if for every $\varepsilon > 0$ there exist $\mathcal{B}_d$-interval functions $F_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2 \in \underline{\mathcal{A}}_{\mathcal{B}}$ such that

(A) for each $\mathbf{x}$ with all dyadic-irrational coordinates

$$\underline{D}_{\mathcal{B}} F_1(\mathbf{x}) \geq f(\mathbf{x}) \geq \overline{D}_{\mathcal{B}} F_2(\mathbf{x});$$

(B) F_1 and F_2 are continuous in the sense of Saks everywhere on $[0,1]^m$;

(C) $F_1([0,1]^m) - F_2([0,1]^m) < \varepsilon$.

$\overline{P}_{\mathcal{B}_d}$*-integral* of the function f on $[0,1]^m$ is defined as

$$(\overline{P}_{\mathcal{B}_d}) \int_I f := \inf_{F_1} F_1([0,1]^m) = \sup_{F_2} F_2([0,1]^m).$$

THEOREM 4.5 *If the series* (4.5) *or* (4.6) *is rectangular convergent to a sum f everywhere on* $[0,1]^m$*, then the function f is* $\overline{P}_{\mathcal{B}_d}$*-integrable and* (4.5) *or* (4.6) *is the Fourier series of f, in the sense of the* $\overline{P}_{\mathcal{B}_d}$*-integral.*

6. Coefficients Problem for Regular Rectangular Convergent Series

Continuity in the sense of Saks can not be used to solve the problem in the case of regular convergence. This is clear from the following result (see [10]).

PROPOSITION 4.4 *For every* $\rho \in (0, 1]$ *there is a double Walsh series* ρ*-regularly convergent to a finite sum everywhere on* $[0, 1]^m$*, but* $\mathcal{B}_d$*-interval function* ψ *constructed for this series by* (4.7) *is not continuous in the sense of Saks.*

We shall use here another type of continuity at the points with dyadic-rational coordinates

In [4] the problem of recovering the coefficients of everywhere convergent double Haar series was considered. In that paper a Perron-type integral was constructed. We introduce a modified version of this integral.

DEFINITION 4.7 We say that a finite function f defined on $[0, 1]^2$ is $(P_d^{1/2})$-*integrable* if for every $\varepsilon > 0$ there exist $\mathcal{B}_d$-interval functions $F_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2 \in \underline{\mathcal{A}}_{\mathcal{B}}$ with the following properties:
(A) if $\mathbf{t} = (t_1, t_2) \in ([0, 1] \setminus Q_d) \times ([0, 1] \setminus Q_d)$ and $\{I_{k_1,k_2}\}$ is the basic sequence convergent to $\mathbf{t}$, then

$$\underline{\lim}_{k\to\infty} \frac{F_1(I_{k,k})}{|I_{k,k}|} \geq f(\mathbf{t}) \geq \overline{\lim}_{k\to\infty} \frac{F_2(I_{k,k})}{|I_{k,k}|};$$

(B) if $\mathbf{t} = (t_1, t_2) \in Q_d \times ([0, 1] \setminus Q_d)$ and $\{I_{k_1,k_2}\}$ is the basic sequence convergent to $\mathbf{t}$, then

$$\lim_{k\to\infty} \frac{1}{|I_{k,k}|}\left(F_i(I_{k,k}) - \frac{1}{2}F_i(I_{k-1,k})\right) = 0 \quad (i = 1, 2);$$

(C) if $\mathbf{t} = (t_1, t_2) \in ([0, 1]^2 \setminus Q_d) \times Q_d$ and $\{I_{k_1,k_2}\}$ be the basic sequence convergent to $\mathbf{t}$, then

$$\lim_{k\to\infty} \frac{1}{|I_{k,k}|}\left(F_i(I_{k,k}) - \frac{1}{2}F_i(I_{k,k-1})\right) = 0 \quad (i = 1, 2);$$

(D) if $\mathbf{t} = (t_1, t_2) \in Q_d \times Q_d$ and $\{I_{k_1,k_2}\}$ be the basic sequence convergent to $\mathbf{t}$, then $\lim_{k\to\infty} \frac{1}{|I_{k,k}|}\Bigg(F_i(I_{k,k}) - \frac{1}{2}F_i(I_{k,k-1}) - \frac{1}{2}F_i(I_{k-1,k}) + \frac{1}{4}F_i(I_{k-1,k-1})\Bigg) = 0 \ \ (i = 1, 2);$
(E) $F_1([0, 1]^2) - F_2([0, 1]^2) < \varepsilon$.

For every dyadic interval $I \subset [0,1]^2$ we define $(P_d^{1/2})$-*integral* of the function f on I as $(P_d^{1/2}) \int_I f(t_1,t_2) := \inf_{F_1} F_1(I) = \sup_{F_2} F_2(I)$.

THEOREM 4.6 (see [4, theorem 2]) *Let* $\rho \in (0,1/2]$ *be chosen. Suppose that a double Haar series* (4.6) ρ*-regularly converges to a finite sum* $f(t_1,t_2)$ *everywhere on* $[0,1]^2$. *Then the function* f *is* $(P_d^{1/2})$*-integrable and* (4.6) *is its Fourier series in the sense of the* $(P_d^{1/2})$*-integral.*

The condition $\rho \in (0,1/2]$ in the last theorem can not be replaced by the condition $\rho = 1$. It is shown in [5] that there exists a non-trivial double Haar series convergent cubically (i.e., 1-regularly) to zero everywhere on $[0,1]^2$.

One of the properties of $(P_d^{1/2})$-integral is that this integral and Lebesgue one are incomparable [6]. But these integrals are compatible, i.e., they do not contradict to each other (see [8]). In [7] the construction of $(P_d^{1/2})$-integral was modified and the family of two-dimensional integrals was constructed. This family solves the coefficients problem for double Haar series if a special subsequence of rectangular partial sums is convergent (see [7, theorem 2]).

In [9] a generalization of $(P_d^{1/2})$-integral was introduced. We present here a modified version of this generalization.

Let Σ_m be the set of m-dimensional vectors $\sigma = (\sigma_1,\dots,\sigma_m)$ with $\sigma_i \in \{0,1\}$ $(i = 1,\dots,m)$. For $\mathbf{t} = (t_1,\dots,t_m) \in [0,1]^m$ we denote by $\Sigma_{\mathbf{t},m}$ the set of m-dimensional vectors $\sigma = (\sigma_1,\dots,\sigma_m)$ with $\sigma_i \in \{0,1\}$ such that if $t_i \in Q_d$, then $\sigma_i = 1$. Let $\{I_{\mathbf{k}}\}$ be a basic sequence of intervals (4.3) convergent to a point $\mathbf{t} \in [0,1]^m$. Put

$$I_{k_i}^0 = I_{k_i+1}, \quad I_{k_i}^1 = I_{k_i} \setminus I_{k_i+1}.$$

If $\sigma \in \Sigma_{\mathbf{t},m}$ or $\sigma \in \Sigma_m$ then we define by $I_{\mathbf{k}}^\sigma$ the dyadic interval $I_{k_1}^{\sigma_1} \times \dots \times I_{k_m}^{\sigma_m}$. By $|\sigma|$ denote the sum $|\sigma_1| + \dots + |\sigma_m|$.

Let $\mathbf{t} \in [0,1]^m$. We say that a function τ is Σ_m*-continuous* at a point $\mathbf{t}$ if the equation

$$\lim_{k_1 = \dots = k_m \to \infty} \sum_{\sigma \in \mathbf{\Sigma_m}} (-1)^{|\sigma|} \tau(I_{\mathbf{k}}^\sigma) = 0$$

holds for any basic sequence $\{I_{\mathbf{k}}\}$ convergent to the point $\mathbf{t}$.

DEFINITION 4.8 Let f be a finite function defined on $([0,1] \setminus Q_d)^m$ except possibly on a countable set L. We say that a function f is $(P_d^{1/2,*})$-*integrable* if for every $\varepsilon > 0$ there exist $\mathcal{B}$-interval functions $F_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2 \in \underline{\mathcal{A}}_{\mathcal{B}}$ with the following properties:
(A) F_1 and F_2 are Σ_m-continuous at every point $\mathbf{t} \in [0,1]^m$;
(B) if $\mathbf{t} \in ([0,1] \setminus Q_d)^m \setminus L$ and $\{I_{\mathbf{k}}\}$ be the basic sequence of the form (4.3)

convergent to $\mathbf{t}$, then

$$\underline{\lim}_{k_1=\ldots=k_m\to\infty}\frac{F_1(I_{\mathbf{k}})}{|I_{\mathbf{k}}|}\geq f(\mathbf{t})\geq\overline{\lim}_{k_1=\ldots=k_m\to\infty}\frac{F_2(I_{\mathbf{k}})}{|I_{\mathbf{k}}|};$$

(C) if a point $\mathbf{t}\in[0,1]^m\setminus L$ has exactly $i\in\{1,\ldots,m\}$ dyadic-rational coordinates and $\{I_{\mathbf{k}}\}$ is the basic sequence convergent to $\mathbf{t}$, then

$$\lim_{k_1=\ldots=k_m\to\infty}\frac{1}{|I_{\mathbf{k}}^{\sigma}|^{1-i/m}}\sum_{\sigma\in\mathbf{\Sigma_{t,m}}}(-1)^{|\sigma|}F_i(I_{\mathbf{k}}^{\sigma})=0\quad(i=1,2);$$

(D) $F_1([0,1]^m)-F_2([0,1]^m)<\varepsilon$.

The $(P_d^{1/2,*})$-*integral* of the function f on a dyadic interval $I\subset[0,1]^m$ is defined as $\inf_{F_1}F_1(I)=\sup_{F_2}F_2(I)$.

THEOREM 4.7 (see [9, theorem 6]). Let $\rho\in(0,1/2]$ be chosen. Suppose that the series (4.6) and some countable set $L\subset[0,1]^m$ satisfy the following conditions:
(1) for any $\mathbf{t}\in[0,1]^m$

$$b_{\mathbf{n}}\chi_{\mathbf{n}}(\mathbf{t})=\overline{\overline{o}}_{\mathbf{t}}(n_1\cdot\ldots\cdot n_m),\quad\min_i\{n_i\}\to\infty,\quad\min_{i,j}\{n_i/n_j\}\geq 1/2;\quad(4.8)$$

(2) for all $\mathbf{t}\in(I_d)^m\setminus L$ the series (4.6) ρ-regularly converges to a finite sum $f(\mathbf{t})$;
(3) if a point $\mathbf{t}\in[0,1]^m\setminus L$ has exactly $i\in\{1,\ldots,m\}$ dyadic-rational coordinates then

$$S_{\mathbf{N}}(\mathbf{t})=\overline{\overline{o}}_{\mathbf{t}}((N_1\cdot\ldots\cdot N_m)^{i/m}),\quad\min_i\{N_i\}\to\infty\quad\min_{i,j}\{N_i/N_j\}\geq 1/2.$$

Then the function f is $(P_d^{1/2,*})$-integrable and 4.6 is its Fourier series in the sense of the $(P_d^{1/2,*})$-integral.

In [10] the group G^m instead of the unit cube $[0,1]^m$ was considered. In this case a more simple integral solving the coefficients problems for both Haar and Walsh series was constructed.

DEFINITION 4.9 Let for every point $\mathbf{t}\in G^m$ except possibly a countable set L an increasing sequence of natural numbers $\{k_j=k_j(\mathbf{t})\}$ be chosen. We say that a finite function f defined on $G^m\setminus L$ is $P(k_j)$-*integrable* if for every $\varepsilon>0$ there exist $\mathcal{B}$-interval functions $F_1\in\overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2\in\underline{\mathcal{A}}_{\mathcal{B}}$ with the following properties:
(A) F_1 and F_2 are Σ_m-continuous at every point $\mathbf{t}\in G^m$;

(B) if $\mathbf{t}$ is any point of $G^m \setminus L$ and $\{I_{\mathbf{k}}\}$ is the basic sequence convergent to $\mathbf{t}$ then

$$\underline{\lim}_{j\to\infty} F_1(I_{k_j,\ldots,k_j})/|I_{k_j,\ldots,k_j}| \geq f(\mathbf{t}) \geq \overline{\lim}_{j\to\infty} F_2(I_{k_j,\ldots,k_j})/|I_{k_j,\ldots,k_j}|;$$

(C) $F_1(G^m) - F_2(G^m) < \varepsilon$.
For every dyadic interval I the $P(k_j)$-*integral* of the function f on I is defined as $\inf_{F_1} F_1(I) = \sup_{F_2} F_2(I)$.

The next theorems were proven in [10].

THEOREM 4.8 *Let at every point $\mathbf{t} \in G^m$, except possibly a countable set L, the increasing sequence of natural numbers $\{k_j = k_j(\mathbf{t})\}$ be chosen. Assume that for the series (S) of the form (4.6) the following conditions hold:*
(1) *at every point $\mathbf{t} \in \mathbf{G^m} \setminus \mathbf{L}$ the subsequence $S_{2^{k_j(\mathbf{t})},\ldots,2^{k_j(\mathbf{t})}}(\mathbf{t})$ of the cubical partial sums of the series (S) converges to a finite sum $f(\mathbf{t})$ as $j \to \infty$;*
(2) *at every point $\mathbf{t} \in G^m$ the series (S) satisfies the condition (4.8).*
Then the function $f(\mathbf{t})$ is $(P(k_j))$-integrable and the series (S) is its Fourier series in the sense of the $(P(k_j))$-integral.

A similar result holds for the Walsh series.

THEOREM 4.9 *Let at every point $\mathbf{t} \in G^m$ except possibly a countable set L an increasing sequence of natural numbers $\{k_j = k_j(\mathbf{t})\}$ be chosen. Assume that for the series (S) of the form (4.5) the following conditions hold:*
(1) *at every point $\mathbf{t} \in G^m \setminus L$ the subsequence $S_{2^{k_j(\mathbf{t})},\ldots,2^{k_j(\mathbf{t})}}(\mathbf{t})$ of the cubical partial sums of the series (S) converges to a finite sum $f(\mathbf{t})$ as $j \to \infty$;*
(2) *the series (S) satisfies the condition*

$$a_{\mathbf{n}} = \overline{\overline{o}}(1), \quad \min_i\{n_i\} \to \infty, \quad \min_{i,j}\{n_i/n_j\} \geq 1/2.$$

Then the function $f(\mathbf{t})$ is $(P(k_j))$-integrable and the series (S) is $(P(k_j))$-Fourier series of the function $f(\mathbf{t})$.

As a corollary we get

THEOREM 4.10 *Let a number $\rho \in (0, 1/2]$ be chosen. Suppose that the m-multiple Walsh or Haar series ρ-regularly converges to a finite sum $f(\mathbf{t})$ at every point $\mathbf{t} \in G^m$ except possibly a countable set L. Then for every choice of a sequence $\{k_j = k_j(\mathbf{t})\}$ the function $f(\mathbf{t})$ is $(P(k_j))$-integrable and the given series is $(P(k_j))$-Fourier series of the function $f(\mathbf{t})$.*

For more details see [7], [8], [10] and [14].

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