Chapter 15

SETS OF UNIQUENESS FOR MULTIPLE WALSH SERIES

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Abstract This brief review is devoted to the uniqueness problem for multiple Walsh series.

1. Introduction

The theory of uniqueness for orthogonal system originated from the well-known Cantor's theorem [1872]:

If a trigonometric series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ converges to zero everywhere on $[0, 2\pi]$, then this series is identically zero, i.e., $a_n = 0$ for all $n \in \mathbf{Z}$.

We remind the principal definitions.

Let $\{\psi_n\}$ be an orthogonal system of functions on some set A. Consider a series

$$\sum_n c_n \psi_n. \tag{15.1}$$

A set $L \subset A$ is called a *set of uniqueness* (or in short: a U*-set*) for the system $\{\psi_n\}$ if from the convergence of the series (15.1) to zero outside the set L it follows that $c_n = 0$ for all n. If a set L is not a U-set for the system $\{\psi_n\}$ then it is called a *set of multiplicity* (or in short: a M*-set*) for the system $\{\psi_n\}$. With this terminology the Cantor theorem states that $\emptyset$ is a U-set for trigonometric series.

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2. Walsh-Paley System

Now we remind some basic results for one-dimensional Walsh-Paley system.

Vilenkin [13] first proved that $\emptyset$ is a U-set (for Walsh series). Šneider [12], and, independently, Fine [3] have shown that every finite or countable set $L \subset [0,1]$ is a U-set. Šneider [12] also proved that every set $L \subset [0,1]$ of positive Lebesgue measure is a M-set and that there are uncountable Borel U-sets. Coury [2] has constructed a Borel M-set whose Lebesgue measure is zero. Skvortsov [11] strengthened the last result. He proved that there is a perfect M-set, whose Hausdorff p-measure is zero for all $p > 0$ (the definition of the Hausdorff p-measure is given below). Wade [14] proved that if $L_1,\ L_2, ...$ is a sequence of closed U-sets then $\bigcup_{n=1}^{\infty} L_n$ is also a U-set.

3. Multidimensional Case

Now we consider the multidimensional case. We write G for the *dyadic group*.

Let $\{\omega_n(t)\}_{n=0}^{\infty}$ be the *Walsh-Paley system* on $[0,1]$ or on G. Fix natural $d \geq 1$. Consider the *d-multiple Walsh system* on the G^d, i.e.,

$$\{\omega_{\mathbf{n}}(\mathbf{t})\} = \{\omega_{n_1}(t^1) \cdot \ldots \cdot \omega_{n_d}(t^d)\}, \quad \mathbf{n} = (n_1, \ldots, n_d), \quad \mathbf{t} = (t^1, \ldots, t^d). \tag{15.2}$$

The *d-multiple Walsh series* is defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} \omega_{\mathbf{n}}(\mathbf{t}) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_d=0}^{\infty} b_{n_1,\ldots,n_d} \prod_{i=1}^{d} \omega_{n_i}(t^i), \tag{15.3}$$

where $b_{\mathbf{n}}$ are real numbers. If $\mathbf{N} = (N_1, \ldots, N_d)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of the series (15.3) at a point $\mathbf{t}$ is

$$S_{\mathbf{N}}(\mathbf{t}) = \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_d=0}^{N_d-1} b_{\mathbf{n}} \omega_{\mathbf{n}}(\mathbf{t}).$$

Let $d \geq 2$; then the series (15.3) *converges rectangularly* to a sum $S(\mathbf{t})$ at a point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \text{ as } \min_i \{N_i\} \to \infty.$$

Let $\rho \in (0,1]$; then the series (15.3) *converges ρ-regularly* to a sum $S(\mathbf{t})$ at a point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \text{ as } \min_i \{N_i\} \to \infty \text{ and } \min_{i,j} \{N_i/N_j\} \geq \rho.$$

The 1-regular convergence we call *cubical*.

It is obvious that if the series (15.3) converges rectangularly to a sum $S(\mathbf{t})$ at a point $\mathbf{t}$ then for every $\rho \in (0,1]$ this series converges ρ-regularly to $S(\mathbf{t})$ at $\mathbf{t}$.

Let $U_{\text{rect},d}$ ($U_{\rho,d}$) denote the class of all U-sets for the system (15.2) with respect to the rectangular (ρ-regular) convergence. Obviously, $U_{\rho,d} \subset U_{\text{rect},d}$ for all $\rho \in (0,1]$.

It follows from results by Skvortsov [11] and Movsisyan [8] that if $L \subset [0,1]^d$ be a countable set, then $L \in U_{\text{rect},d}$. The wide class of $U_{\text{rect},d}$-sets was constructed by Lukomskiĭ [1989]. In this work it was also proved that if $E \subset [0,1]^{d-1}$, then $E \in U_{\text{rect},d-1}$ if and only if $E \times G \in U_{\text{rect},d}$. In particular there exist uncountable Borel sets $L \in U_{\text{rect},d}$. Kholshchevnikova [4] proved the following multidimensional analogue of the Wade result: Let $L_1, L_2, \ldots$ be a sequence of closed $U_{\text{rect},d}$-sets such that $L_n \subset (0,1)^d$ and $\text{mes} L_n = 0$. Then $\bigcup_{n=1}^{\infty} L_n$ is a $U_{\text{rect},d}$-set.

Below the series (15.3) are considered on G^d instead of on $[0,1]^d$. Lukomskiĭ [1996] proved that $\emptyset \in U_{\rho,d}$ for all $\rho \in (0,1]$. In Lukomskiĭ [6] it was proven that $U_{\rho,d} \neq U_{\text{rect},d}$ for all $\rho \in (0,1]$.

We have proved the following results (see Plotnikov [9] and [10]).

THEOREM 15.1 *Let $L \subset G^d$ be a finite set. Then, $L \in U_{1,d}$.*

THEOREM 15.2 *There exist countable sets $L \in U_{1,d}$.*

THEOREM 15.3 *Let $L \subset G^d$ be a countable set. Then, $L \in U_{1/2,d}$.*

Next we use the concepts of the Hausdorff p-measure and the Hausdorff dimension. Let $A \subset \mathbf{R}^d$. *A covering* of a set A is a collection $I = \{\Delta\}$ of sets $\Delta \subset \mathbf{R}^d$ such that $A \subset \bigcup_{\Delta \in I} \Delta$. Then *the Hausdorff p-measure* of a set A is defined by

$$\text{mes}_p A = \liminf_{\varepsilon \to +0} \sum (\text{diam}\, \Delta)^p, \tag{15.4}$$

where $\{\Delta\}$ is a finite or countable covering of a set A by cubes or balls Δ with $\text{diam}\, \Delta < \varepsilon$.

By *the Hausdorff dimension* of a set $A \subset \mathbf{R}^d$ we mean the number $\dim_H A$ equal to

$$\sup\{p \in \mathbf{R},\ \text{mes}_p A > 0\}. \tag{15.5}$$

Define the Hausdorff dimension of a set $L \subset G^d$. Recall that the dyadic group G is a set of the infinite sequences $t = \{t_k\}$ where $t_k = 0$ or 1. Consider the map $\varphi : G \to [0,1]$ given by the formula

$$\varphi(t) = \sum_{k=0}^{\infty} t_k 2^{-k-1}.$$

Let $\phi(\mathbf{t}) = (\varphi(t^1), \ldots, \varphi(t^d))$ for $\mathbf{t} = (t^1, \ldots, t^d)$. Using (15.4) and (15.5) we define the Hausdorff dimension of a set $L \subset G^d$ by

$$\dim_H L = \dim_H(\phi(L)), \quad (\phi(L) \subset \mathbf{R}^d).$$

THEOREM 15.4 *For every $d = 1, 2, \ldots$ there is a perfect set $L \in U_{1,d}$ such that $\dim_H L = d$.*

In the case $d = 1$ the Theorem 15.4 and the results by Skvortsov [1977] are the complement of one another. These theorems show that the property of a set to be a U-set or a M-set does not depend on the Hausdorff dimension of the set.

4. Open Questions

In conclusion we state four open questions which seem to be of interest.

Question 1. *Let $\rho \in (0, 1]$ be chosen. Consider the series (15.3) on $[0, 1]^d$. Is $\emptyset \in U_{\rho,d}$? The similar problem is also open for the multiple trigonometric system.*

Question 2. *Is any set $L \subset [0, 1]^d$ $($resp. $L \subset G^d)$ with positive Lebesgue measure $($resp. Haar measure$)$ a set of multiplicity of d-multiple Walsh system for the rectangular or ρ-regular convergence? The analogous problem is open for the multiple trigonometric system.*

Question 3. *Is any countable set $L \subset G^d$ a set of uniqueness for d-multiple Walsh system for the cubical convergence?*

Question 4. *Consider the series (15.3) on G^d. Are there $0 < \rho_1 < \rho_2 \leq 1$ such that $U_{\rho_1,d} \neq U_{\rho_2,d}$? The similar problem is open for the multiple trigonometric system.*

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