Chapter 7

WAVELET LIKE TRANSFORM ON THE BLASCHKE GROUP

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Abstract Blaschke-functions play an important role in system identification. These functions form a group with respect to the composition of functions. In the papers [5],[6],[7] we introduced a new transform connected to this group. It is in same relation with this group as the affine wavelet transform with the affine group, or the Gábor-transform with the Heisenberg-group. In this paper we give a summary of the main concepts and results. The first section contains the basic notations, definitions and results connected to the representations and the voice transform.

In section 2 the voice transform of the Blaschke group is studied on the Lebesgue $L^2(\mathbf{T})$ space and on the Hardy space $H^2(\mathbb{D})$. The discrete Laguerre system can be obtained as wavelet system with respect to the Blaschke group. In section 3 we consider the voice transform on the Bergman space.

1. The Voice Transform

In signal processing and image reconstruction the Fourier-, wavelet-, Gábor-transforms play important roles. There exists a common generalization of these transformations, the so-called *Voice-transformation*. In this section we summarize the basic notions used in the definition of Voice-transform. We also present the definition and the most important properties of this transform.

In the construction of Voice-transform the starting point will be a locally compact topological group $(G, \cdot)$. It is known that every locally compact topological group has nontrivial left- and right-translation invariant Borel-measures,

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called left invariant and right invariant *Haar-measures.* Let m be a nontrivial left-invariant Haar measure of G, and let $f : G \to \mathbb{C}$ be a Borel-measurable function integrable with respect to the m. The integral of f will be denoted by $\int_G f\,dm = \int_G f(x)\,dm(x)$. Because of the left-translation invariance of the measure m it follows that

$$\int_G f(x)\,dm(x) = \int_G f(a^{-1}\cdot x)\,dm(x) \ \ (a \in G).$$

There exist groups whose left invariant Haar measure is no right invariant. If the left invariant Haar measure of G is in the same time right invariant then G is called *unimodular group*. On a given group, Haar measure is unique only up to constant multiples. It is trivial that the commutative groups are unimodular. Furthermore it can be proved that if the left Haar measure is invariant under the inverse transformation $G \ni x \to x^{-1} \in G$, then G is also unimodular (for details see [8],[9]).

In the definition of the voice-transform a *unitary representation of the group* $(G,\cdot)$ is used. Let us consider a Hilbert-space $(H, \langle\cdot,\cdot\rangle)$ and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear bijections which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group, the neutral element of which is I, the identity operator on H. The inverse element of $U \in \mathcal{U}$ is the operator U^{-1}. It is equal to the adjoint operator U^*. The homomorphism of the group $(G,\cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying

$$\begin{aligned} &\text{i)} \quad U_{x\cdot y} = U_x \circ U_y \ \ (x, y \in G), \\ &\text{ii)} \quad G \ni x \to U_x f \in H \ \ \text{is continuous for all } f \in H \end{aligned} \tag{7.1}$$

is called the unitary representation of $(G,\cdot)$ on H. The *voice transform* of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex-valued) function on G defined by

$$(V_\rho f)(x) := \langle f, U_x\rho\rangle \quad (x \in G, f, \rho \in H). \tag{7.2}$$

For any representation $U : G \to \mathcal{U}$ and for each $f, \rho \in H$ the voice transform $V_\rho f$ is a continuous and bounded function on G.

The set of continuous bounded functions defined on the group G with the usual norm $\|F\| := \sup\{|F(x)| : x \in G\}$, form a Banach space. From the unitarity of $U_x : H \to H$ follows that, for all $x \in G$

$$|(V_\rho f)(x)| = |\langle f, U_x\rho\rangle| \le \|f\|\|U_x\rho\| = \|f\|\|\rho\|,$$

consequently V_ρ is linear, bounded and $\|V_\rho\| \le \|\rho\|$.

Taking as starting point (not necessarily commutative) locally compact groups and a unitary representation we can construct important transformations in signal processing and control theory in this way. For example the Fourier transform, the affine wavelet transform and the Gábor-transform are all special voice transforms (see [4], [8]).

The invertibility of the voice transform V_ρ is connected to the irreducibility of the representation U. The representation U is called *irreducible* if the only closed invariant subspaces of H, i.e. the closed subspaces H_0 which satisfy $U_x H_0 \subset H_0$ $(x \in G)$, are $\{0\}$ and H. Since the closure of the linear span of the set

$$\{U_x \rho : x \in G\} \tag{7.3}$$

is always a closed invariant subspace of H, it follows that U is irreducible if and only if the collection (1.3) is a closed system for any $\rho \in H, \rho \neq 0$.

The property of irreducibility gives a simple criterion for deciding when a voice transform is $1-1$ (see [4], [8]):

I. *A voice transform V_ρ generated by a unitary representation U is $1-1$ for all $\rho \in H \setminus \{0\}$ if and only if U is irreducible.*

The function $V_\rho f$ is continuous on G but in general is not square integrable. If there exists $\rho \in H, \rho \neq 0$ such that $V_\rho \rho \in L^2_m(G)$, then the representation U is called *square integrable* and ρ is called admissible for U. For a fixed square integrable U the collection of admissible elements of H will be denoted by H^*. Choosing a convenient $\rho \in H^*$ the voice transform $V_\rho : H \to L^2_m(G)$ will be unitary. This is consequence of the following claim (see [4], [8]):

II. *Let $(U_a)_{a \in G}$ be an irreducible square integrable representation of the group G on the Hilbert space H. The collection H^* of admissible elements is a linear subspace of H and for every $\rho \in H^*$ the voice transform of the function f is square integrable on G, namely $V_\rho f \in L^2_m(G)$, if $f \in H$. Moreover there is a symmetric, positive bilinear map $B : H^* \times H^* \to \mathbb{R}$ such that*

$$[V_{\rho_1} f, V_{\rho_2} g] = B(\rho_1, \rho_2)\langle f, g\rangle \quad (\rho_1, \rho_2 \in H^*, f, g \in H), \tag{7.4}$$

where $[\cdot,\cdot]$ is the usual inner product in $L^2_m(G)$. If the group G is unimodular then $B(\rho, \rho) = c\langle \rho, \rho\rangle$ $(\rho \in H^*)$, where $c > 0$ is a constant. In this case if we choose ρ so that $\langle \rho, \rho\rangle = 1/c$ then

$$[V_\rho f, V_\rho g] = \langle f, g\rangle \quad (f, g \in H). \tag{7.5}$$

In the next sections we will construct voice transform using so called *multiplier representations* generated by a collection of multiplier functions defined in the following way: $F_a : G \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ $(a \in G)$ is a *collection of multiplier functions* if

$$F_e = 1, \;\; F_{a_1 \cdot a_2}(x) = F_{a_1}(a_2 \cdot x) F_{a_2}(x) \;\; (a_1, a_2, x \in G), \tag{7.6}$$

where e is the neutral element of G. It can be proved that

$$(U_a f)(x) := F_{a^{-1}}(x) f(a^{-1} \cdot x) \ \ (a, x \in G)$$

satisfies

$$U_{a_1}(U_{a_2} f) = U_{a_1 \cdot a_2} f \quad (a_1, a_2 \in G),$$

so it is a representation of G on the space of all complex valued functions on G. If F_a is bounded and continuous on G for every $a \in G$, then $L^2_m(G)$ is an invariant subspace and $(U_a)_{a \in G}$ is a representation on G. The representations obtained as bellow are named *multiplier representations* (see [9]). Let us denote by L_a the left translation by a^{-1}, i.e., for any function $f : G \to \mathbb{C}$ set

$$(L_a f)(x) := f(a^{-1} \cdot x). \tag{7.7}$$

It can be proved that the voice transform and the left translation operator satisfies the following properties (see [4], [8]):

III. *Let $(U_a)_{a \in G}$ be a unitary multiplier representation of G which is generated by $F_a \in C(G) \cap L^\infty_m(G)$ $(a \in \mathbb{G})$. Then*

$$\begin{aligned}
&\text{i)} \quad V_\rho \circ U_a = L_a \circ V_\rho, \\
&\text{ii)} \quad V_\rho \circ L_a = L_a \circ V_\rho \circ M_a \\
&\text{iii)} \quad (V_\rho f)(x) = (V_f \rho)(x^{-1}) \quad (a, x \in G, \rho \in H),
\end{aligned}$$

where M_a denotes the multiplication by $F_{a^{-1}}$.

We note that i) and iii) are valid for all unitary representations.

2. The Voice Transforms on the Blaschke Group

The affine wavelet transform is a voice transform of the affine group which is a subgroup of the Möbius group (i.e. the group of linear fractional transformations with the composition operation). In this section we will study the voice transforms of another subgroup of the Möbius group, namely the voice transforms of the Blaschke group.

The Blaschke group

The so called *Blaschke functions* are defined as

$$B_a(z) := \varepsilon \frac{z - b}{1 - \bar{b} z} \ \ (z \in \mathbb{C}, a = (b, \varepsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}), \tag{7.8}$$

where

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \ \ \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}. \tag{7.9}$$

If $a \in \mathbb{B}$, then B_a is an 1-1 map on $\mathbb{T}$ and $\mathbb{D}$ a respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_a, a \in \mathbb{B}\}, \circ)$.

If we use the notations $a_j := (b_j, \varepsilon_j),\ j \in \{1, 2\}$ and $a := (b, \varepsilon) =: a_1 \circ a_2$ then

$$\begin{aligned} b &= \frac{b_1\overline{\varepsilon}_2 + b_2}{1 + b_1\overline{b}_2\overline{\varepsilon}_2} = B_{(-b_2,1)}(b_1\overline{\varepsilon}_2), \\ \varepsilon &= \varepsilon_1 \frac{\varepsilon_2 + b_1\overline{b}_2}{1 + \varepsilon_2\overline{b}_1 b_2} = B_{(-b_1\overline{b}_2,\varepsilon_1)}(\varepsilon_2). \end{aligned} \tag{7.10}$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \varepsilon) \in \mathbb{B}$ is $a^{-1} = (-b\varepsilon, \overline{\varepsilon})$.

Since $B_a : \mathbb{T} \to \mathbb{T}$ is a bijection it follows the existence of $\beta_a > \mathbb{R} \to \mathbb{R}$ such that $B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R})$, where β_a can be expressed in a explicit form. Namely, let us introduce the function

$$\gamma_r(t) := \int_0^t \frac{1 - r^2}{1 - 2rcoss + r^2}\, ds \ \ (t \in \mathbb{R}, 0 \le q < r). \tag{7.11}$$

Then

$$\begin{aligned} \beta_a(t) &:= \theta + \varphi + \gamma_{s(r)}(t - \varphi), \\ s &:= s(r) := \frac{1 + r}{1 - r} \ \ (a = (re^{i\varphi}, e^{i\theta}) \in \mathbb{B}, t \in \mathbb{R}). \end{aligned} \tag{7.12}$$

The group $(\mathbb{B}, \cdot)$ is unimodular and the integral of the function $f : \mathbb{B} \to \mathbb{C}$, with respect to the Haar-measure m of the group $(\mathbb{B}, \circ)$ can be written in the form

$$\int_{\mathbb{B}} f(a)\, dm(a) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2}\, db_1 db_2 dt, \tag{7.13}$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

We will study the voice transform of the Blaschke-group. In the construction there will be used a class of unitary representations of the Blaschke-group on the Hilbert space $H = L^2(\mathbb{T})$.

The voice transform on $L^2(\mathbb{T})$

In this section the voice transform on the Hilbert space $H = L^2(\mathbb{T})$ will be constructed, where the inner product is given by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{I}} f(e^{it})\overline{g(e^{it})}\,dt \ \ (f, g \in H).$$

The trigonometric system $\varepsilon_n(t) = e^{int}(t \in \mathbb{I}, n \in \mathbb{Z})$ is orthonormal and complete with respect this scalar product.

It can be proved that the function

$$F_a(e^{it}) := \sqrt{\beta'_a(t)}e^{i(\beta_a(t)-t)/2} \ (a \in \mathbb{B}, t \in \mathbb{I})$$

is a multiplier. The representation of the Blaschke-group on $L^2(\mathbb{T})$ generated by this multiplier is given by

$$\begin{aligned}(U_a f)(e^{it}) &:= F_{a^{-1}}(e^{it}) \cdot f \circ \beta_{a^{-1}}(t) \\ &= f(e^{i\beta_{a^{-1}}(t)})(\beta'_{a^{-1}}(t))^{1/2}e^{i(\beta_{a^{-1}}(t)-t)/2} \ (a \in \mathbb{B}).\end{aligned} \tag{7.14}$$

It can be proved (see [5], [6])

IV. *The representation* U_a $(a \in \mathbb{B})$ *given by the formula (7.14) is a unitary representation of the Blaschke-group on* $L^2(\mathbb{T})$. Denote by $H^2(\mathbb{T})$ the closure in $L^2(\mathbb{T})$-norm of the set span$\{\varepsilon_n, n \in \mathbb{N}\}$. The functions which belong to $H^2(\mathbb{T})$ can be obtained as boundary limits of the functions from Hardy-space $H^2(\mathbb{D})$ (see [3]). The restriction of the representation (7.14) on $H^2(\mathbb{T})$ can be considered as a representation on the Hilbert-space $H^2(\mathbb{D})$ and can be given in the form

$$\begin{aligned}(U_{a^{-1}}f)(z) &:= \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\bar{b}z)} f\big(\frac{e^{i\theta}(z-b)}{1-\bar{b}z}\big) \\ &\qquad \big(z = e^{it} \in \mathbb{T}, a = (b, e^{i\theta}) \in \mathbb{B}\big).\end{aligned} \tag{7.15}$$

The voice transform generated by $(U_a)_{a\in\mathbb{B}}$ is given by the following formulae

$$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle \ \ (f, \rho \in H^2(\mathbb{T})). \tag{7.16}$$

Let consider the shift operator

$$(S\varphi)(z) = z\varphi(z), \quad \varphi \in H^2(\mathbb{T}),$$

and let $\varphi = 1 \in H^2(\mathbb{T})$ be the *mother wavelet* (compare [2], [8]). Then the *discrete Laguerre* functions can be generated by the shift operator and by the representation operator in the following way (see [1]) :

$$\begin{aligned} \varphi_{a,m}(z) &:= (U_{a^{-1}}S^m\varphi)(z) \\ &= U_{a^{-1}}\varepsilon_m = \frac{\sqrt{\varepsilon(1-|b|^2)}}{(1-\bar{b}z)}\left(\frac{\varepsilon(z-b)}{1-\bar{b}z}\right)^m . \end{aligned} \tag{7.17}$$

It is known that $(\varphi_{a,m}, m \in \mathbb{N})$ forms an orthogonal basis in $L^2(\mathbb{T})$, for all $a \in \mathbb{B}$ (see [1]).

Let $V_{\varepsilon_m}f(a^{-1}) = \langle f, U_{a^{-1}}\varepsilon_m\rangle$ and let the projection operator Pf defined as

$$Pf(a,z) := \sum_{m=0}^{\infty} V_{\varepsilon_m}f(a^{-1})U_{a^{-1}}\varepsilon_m(z) \ \ (a \in \mathbb{B},\, z \in \mathbb{D}). \tag{7.18}$$

V. *For every $f \in H^2(\mathbb{T})$, $z = r_1e^{it} \in \mathbb{D}$, and $a \in \mathbb{B}$*

$$\lim_{r_1 \to 1} Pf(a,z) = f(e^{it}) \tag{7.19}$$

for ae. $t \in \mathbb{I}$ and in H^2 norm. If $f \in C(\mathbb{T})$, then the convergence is uniform (see [5]).

If $a = e = (0,1)$ then $V_{\varepsilon_m}f(e) = \langle f, \varepsilon_m\rangle$ and $U_{e^{-1}}\varepsilon_m(z) = \varepsilon_m$. Consequently in this special case (7.19) is the analogue of Abel summation for the trigonometric series.

3. The Voice Transform on the Bergman Space

In this section we introduce the voice transform by using a class of unitary representations of the Blaschke group on *Bergman spaces* (see [3]).

Let the set of analytic function $f : \mathbb{D} \to \mathbb{C}$ denoted by $\mathcal{A}$. For $m \in \mathbb{N}, m \geq 2$ let us consider the following subset of analytic functions:

$$\mathcal{B}^m(\mathbb{D}) := \{f \in \mathcal{A} : \frac{1}{2\pi}\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{m-2}\, dxdy < \infty \ \ (z = x+iy \in \mathbb{D})\}.$$

The set $\mathcal{B}^m(\mathbb{D})$ with the scalar product

$$\langle f, g\rangle_m := \frac{1}{2\pi}\int_{\mathbb{D}} f(z)\overline{g(z(z)}\,(1-|z|^2)^{m-2}\, dxdy \ \ (z = x+iy \in \mathbb{D})$$

is a Hilbert space. In the special case when $m = 2$, $\mathcal{B}^2(\mathbb{D})$ is the *Bergman space.* It can be proved that the function

$$f(z) := \sum_{n=0}^{\infty} c_n z^n \ \ (z \in \mathbb{D})$$

from $\mathcal{A}$ belongs to the set $\mathcal{B}^m(\mathbb{D})$ if and only if the coefficients of f satisfy

$$\sum_{n=0}^{\infty} |c_n|^2 \lambda_n^{[m]} < \infty,$$

where

$$\lambda_n^{[m]} := \int_0^1 (1-r^2)^{m-2} r^{2n+1}\, dr \;\; (m \geq 2, n \in \mathbb{N}).$$

When $m = 2$, then

$$\lambda_n^{[2]} := \frac{1}{2(n+1)} \;\; (n \in \mathbb{N}).$$

The Hardy space $H^2(\mathbb{D})$ is a subspace of the Bergman-space $\mathcal{B}^2(\mathbb{D})$.

Let us consider the collection of functions

$$F_a(z) := \frac{\sqrt{\varepsilon(1-|b|^2)}}{1-\overline{b}\, z} \;\; (a = (b,\varepsilon) \in \mathbb{B}, z \in \overline{\mathbb{D}}).$$

For every m $(m \geq 2, m \in \mathbb{N})$ the collection $(F_a)_{a\in\mathbb{B}}$ induces a unitary representation of the Blaschke group on the space $\mathcal{B}^m(\mathbb{D})$. Namely, let us define

$$U_a^{[m]} := F_{a^{-1}}^m\, f \circ B_a^{-1} \;\; (a \in \mathbb{B}, m \in \mathbb{N}, m \geq 2, f \in \mathcal{B}^m(\mathbb{D}).$$

Then the following claim can be proved (see [6], [7]):

VI. *For all $m \in \mathbb{N}, m \geq 2$ $(U_a^{[m]})_{a\in\mathbb{B}}$ is a unitary, irreducible representation of the group $\mathbb{B}$ on the Hilbert space $\mathcal{B}^m(\mathbb{D})$.*

For the Voice-transform V_ρ the following analogue of the Plancherel formula holds.

VII. *The Voice transform induced by $(U_a)_{a\in\mathbb{B}}$ satisfies*

$$[V_{\rho_1} f, V_{\rho_2} g] = \frac{1}{2}\langle \rho_1, \rho_2\rangle \langle f, g\rangle \;\; (\rho_1, \rho_2, f, g \in \mathcal{B}^2(\mathbb{D})),$$

where

$$[F, G] := \int_{\mathbb{B}} F(a)\overline{G(a)}\, dm(a),$$

and m is the (7.13) Haar-measure of the group $\mathbb{G}$. Furthermore every $\rho \in \mathcal{H}^2(\mathbb{D})$ is admissible.

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