Chapter 17

AN INVERSION FORMULA FOR THE MULTIPLICATIVE INTEGRAL TRANSFORM

Valentin Skvortsov *
Department of Mathematics, Moscow State University
Moscow 119991, Russia, and
Institute of Mathematics, Casimirus the Great University
pl. Weyssenhoffa 11, 85-079 Bydgoszcz, Poland
vaskvor2000@yahoo.com

Francesco Tulone
Department of Mathematics ed Applications, University of Palermo
via Archirafi 34, 90123 Palermo, Italy
tulone@math.unipa.it

Abstract An inversion formula for multiplicative integral transform with a kernel defined by characters of a locally compact zero-dimensional abelian group is obtained.

1. Introduction

In the classical trigonometrical case it is known (see [12]) that if the integral

$$\int_{-\infty}^{\infty} a(x)e^{ixy}dx,$$

where $a(x)$ is locally summable, converges everywhere to a function $f(y)$ which is finite and locally summable, then the function $a(x)$ can be recovered by the following inversion formula:

$$a(x) = (C,1)\text{-}\lim_{h\to\infty} \frac{1}{2\pi}\int_{-h}^{h} f(y)e^{-ixy}dy \;\text{ for almost all }\; x$$

*Supported by RFFI-08-01-00286.

(here by $(C,1)$-limit of a function $\phi(h)$ defined on $(0,\infty)$, as $h \to \infty$, we mean $\lim_{h\to\infty} h^{-1}\int_0^h \phi(t)dt$).

A similar problem for the case in which the real line is replaced by a locally compact zero-dimensional abelian group and the kernel of the corresponding integral transform is defined by characters of this group was considered in [6]. Transforms of this kind are usually called multiplicative transforms (see [1]).

In this paper we consider transforms convergent to locally summable functions and in this case we obtain a generalization of the result of [6] by weakening the assumption on the type of convergence of those transforms.

The problem of getting an inversion formula for integral transform is a continual analogue of that of recovering the coefficients of a convergent series with respect to characters of compact zero-dimensional abelian group considered for example in [7].

In comparison with [4] and [6] we consider here transforms directly on the group instead of using a mapping of this group on the real line. That mapping was connected with introduction of a certain ordering in this group.

An advantage of the present new approach is that it permits to obtain some more general results on the coefficient problem and on the inversion formula which do not depend on a particular numeration (the so called Vilenkin-Palley numeration) or, respectively, on ordering of characters, as it was the case in the previous papers.

Our technique uses the notion of derivation basis and Henstock method of computing the integrals.

2. Preliminaries

Let G be a zero-dimensional locally compact abelian group which satisfies the second countability axiom. We suppose also that the group G is periodic. It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$\dots. \supset G_{-n} \supset \dots \supset G_{-2} \supset G_{-1} \supset G_0 \supset G_1 \supset G_2 \dots \supset G_n \supset \dots \quad (17.1)$$

with $G = \bigcup_{n=-\infty}^{+\infty} G_n$ and $\{0\} = \bigcap_{n=-\infty}^{+\infty} G_n$. The subgroups G_n are clopen sets with respect to this topology. As G is periodic, the factor group G_n/G_{n+1} is finite for each n and this implies that G_n (and so also all its cosets) is compact. Note that the factor group G_n/G_0 is also finite for any $n < 0$ and so the factor group G/G_0 is countable. We denote by K_n any coset of the subgroup G_n and by $K_n(g)$ the coset of the subgroup G_n which contains the element g, i.e.,

$$K_n(g) = g + G_n. \quad (17.2)$$

For each $g \in G$ the sequence $\{K_n(g)\}$ is decreasing and $\{g\} = \bigcap_n K_n(g)$.

Now for each coset K_n of G_n we choose and fix for the rest of the paper, an element g_{K_n}. Then for each $n \in \mathbf{Z}$ we can represent any element $g \in G$ in the form:

$$g = g_{K_n} + \{g\}_n \tag{17.3}$$

where $\{g\}_n \in G_n$. We agree to put $g_{G_n} = 0$, so that $g = \{g\}_n$ if $g \in G_n$.

Let Γ denote the dual group of G, i.e., the group of characters of the group G. It is known (see [1]) that under assumption imposed on G the group Γ is also a periodic locally compact zero-dimensional abelian group (with respect to the pointwise multiplication of characters) and we can represent it as a sum of increasing sequence of subgroups

$$.... \supset \Gamma_{-n} \supset ... \supset \Gamma_{-2} \supset \Gamma_{-1} \supset \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset ... \supset \Gamma_n \supset ... \tag{17.4}$$

introducing a topology in Γ. Then $\Gamma = \bigcup_{i=-\infty}^{+\infty} \Gamma_i$ and $\bigcap_{i=-\infty}^{+\infty} \Gamma_i = \{\gamma^{(0)}\}$ where $(g, \gamma^{(0)}) = 1$ for all $g \in G$ (here and below (g, γ) denote the value of a character γ at a point g). For each $n \in \mathbf{Z}$ the group Γ_{-n} is the annulator of G_n, i.e.,

$$\Gamma_{-n} = G_n^{\perp} := \{\gamma \in \Gamma : (g, \gamma) = 1 \text{ for all } g \in G_n\}.$$

The representation (17.3), properties of a character and the definition of the annulator imply

$$(g, \gamma) = (g_{K_n}, \gamma)(\{g\}_n, \gamma) = (g_{K_n}, \gamma).$$

So with a fixed element g_{K_n}, the value (g, γ) is constant for all $g \in K_n$ and we get the following

LEMMA 17.1 *If $\gamma \in \Gamma_{-n}$ then γ is constant on each coset K_n of G_n.*

The factor groups $\Gamma_{-n-1}/\Gamma_{-n} = G_{n+1}^{\perp}/G_n^{\perp}$ and G_n/G_{n+1} are isomorphic (see [1]) and so they are of finite order for each $n \in \mathbf{Z}$. This implies that the group Γ_{-n}/Γ_0 is also finite for any $n > 0$ and Γ/Γ_0 is countable.

Now, as we have done above for the group G, we choose and fix an element $\gamma_J \in J$ for each coset J of Γ_0. Then we can represent any element $\gamma \in \Gamma$ in the form:

$$\gamma = \gamma_J \cdot \{\gamma\} \tag{17.5}$$

where $\{\gamma\} \in \Gamma_0$. We agree to put $\gamma_{\Gamma_0} = \gamma^{(0)}$, so that $\gamma = \{\gamma\}$ if $\gamma \in \Gamma_0$.

We denote by μ_G and μ_Γ the Haar measures on the groups G and Γ, respectively, and we normalize them so that $\mu_G(G_0) = \mu_\Gamma(\Gamma_0) = 1$. We can make these measures to be complete by including all the subsets of the sets of measure zero into the respective class of measurable sets.

The following property of the functions γ can be easily checked (see [8]).

LEMMA 17.2 *If* $\gamma \in \Gamma \setminus \Gamma_{-n}$ *then* $\int_{K_n}(g,\gamma)d\mu_G = 0$ *for each coset* K_n.

It follows from this lemma that if γ_1 and γ_2 are not equal identically on K_n, then they are orthogonal on K_n, i.e.,

$$\int_{K_n}(g,\gamma_1\overline{\gamma_2})d\mu_G = 0.$$

3. Integration on the Group

Now we introduce a construction of an integral on the group which covers the Lebesgue integration with respect to Haar measure on the group. This construction is based on Henstock approach to integration and on the notion of the derivation basis. To define such a derivation basis on the group G, which we denote $\mathcal{B}_G$, we take any function $\nu : G \to \mathbf{Z}$ and define a basis set by

$$\beta_\nu = \{(I,g) : g \in G, I = K_n(g), n \geq \nu(g)\}.$$

Then our basis $\mathcal{B}_G$ is the family $\{\beta_\nu\}_\nu$ where ν runs over the set of all integer-valued functions on G. In the terminology of derivation basis theory any coset K_n, $n \in \mathbf{Z}$, can be called $\mathcal{B}_G$-*interval*. We denote by $\mathcal{I}_G$ the set of all $\mathcal{B}_G$-intervals.

This basis has all the usual properties of a general derivation basis (see [9], [3]). First of all it has the *filter base property*:

- $\emptyset \notin \mathcal{B}_G$,
- for every $\beta_{\nu_1}, \beta_{\nu_2} \in \mathcal{B}_G$ there exists $\beta_\nu \in \mathcal{B}_G$ such that $\beta_\nu \subset \beta_{\nu_1} \cap \beta_{\nu_2}$ (it is enough to take $\nu = \max\{\nu_1, \nu_2\}$).

DEFINITION 17.1 *A* β_ν*-partition is a finite collection* π *of elements of* β_ν*, where the distinct elements* (I', x') *and* (I'', x'') *in* π *have* I' *and* I'' *disjoint.*

If L *is a* $\mathcal{B}_G$*-interval and* $\bigcup_{(I,x)\in\pi} I = L$ *then* π *is called* β_ν*-partition of* L.

Our basis $\mathcal{B}_G$ has the *partitioning property.* It means that the following conditions hold:

- for each finite collection $I_0, I_1, \ldots, I_n$ of $\mathcal{B}_G$-intervals with $I_1, \ldots, I_n \subset I_0$ and $I_i,\ i = 1, 2, \ldots,$ being disjoint, the difference $I_0 \setminus \bigcup_{i=1}^n I_i$ can be expressed as a finite union of pairwise disjoint $\mathcal{B}_G$-intervals;
- for each $\mathcal{B}_G$-interval L and for any $\beta_\nu \in \mathcal{B}_G$ there exists a β_ν-partition of L.

This property of $\mathcal{B}_G$ follows easily from compactness of any $\mathcal{B}_G$-interval and from the fact that any two $\mathcal{B}_G$-intervals I' and I'' are either disjoint or one of them is contained in the other one.

Note that in the case of our basis $\mathcal{B}_G$, given a point $x \in X$, any β_ν-partition contains only one pair (I, x) with this point x.

The following Henstock-type integral was defined in [8]:

DEFINITION 17.2 *Let $L \in \mathcal{I}_G$. A complex-valued function f on L is said to be Kurzweil-Henstock integrable with respect to basis $\mathcal{B}_G$ (or H_G-integrable)* on L, with *H_G-integral A, if for every $\varepsilon > 0$, there exists a function $\nu : L \mapsto \mathbf{Z}$ such that for any β_ν-partition π of L we have:*

$$\left| \sum_{(I,g)\in\pi} f(g)\mu_G(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_G)\int_L f$.

It is easy to check, that a function which is equal to zero almost everywhere on $L \in \mathcal{I}_G$, is H_G-integrable on L with value zero. This justifies the following extension of Definition 17.2 to the case of functions defined only almost everywhere on L.

DEFINITION 17.3 *A complex valued function f defined almost everywhere on $L \in \mathcal{I}_G$ is said to be H_G-integrable on L, with integral value A, if the function*

$$f_1(g) := \begin{cases} f(g), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise} \end{cases}$$

is H_G-integrable on L to A in the sense of Definition 17.2.

It is clear that a complex-valued function is H_G-integrable if and only if its real and imaginary parts are H_G-integrable.

REMARK 17.1 *We note that all the above definitions depend on the structure of the sequence of subgroups (17.1). So if we consider for the group Γ the definitions of the $\mathcal{B}_\Gamma$-basis and the H_Γ-integral, then we should use the sequence (17.4) in our construction.*

The *upper* and the *lower $\mathcal{B}_G$-derivative* of a set function $F : \mathcal{I}_G \mapsto I\!R$ at a point g are defined as

$$\overline{D}_G F(g) := \limsup_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}\,, \quad \underline{D}_G F(g) := \liminf_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{17.6}$$

The *$\mathcal{B}_G$-derivative* at g is

$$D_G F(g) := \lim_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{17.7}$$

It is clear that a real-valued function F is $\mathcal{B}_G$-differentiable if and only if $\overline{D}_G F(x) = \underline{D}_G F(x)$. In this case this common value is equal to $D_G F(x)$. For a complex-valued set function $F = \mathrm{Re}F + i\mathrm{Im}F$ the D_G-derivative at a point x can be defined either directly by (17.7) or as $D_G F(x) = D_G \mathrm{Re}F(x) + iD_G \mathrm{Im}F(x)$.

We say that a set function F is $\mathcal{B}_G$*-continuous* at a point g, with respect to the basis $\mathcal{B}_G$, if $\lim_{n\to\infty} F(K_n(g)) = 0$.

We note that if f is H_G-integrable on $L \in \mathcal{I}_G$ then it is H_G-integrable also on any $\mathcal{B}_G$-subinterval J of L. It can be proved that the indefinite H_G-integral on $L \in \mathcal{I}_G$ is an additive $\mathcal{B}_G$-continuous function on the set of all $\mathcal{B}_G$-subintervals of L.

The property of differentiation of the indefinite H_G-integral almost everywhere was proved in [8]:

THEOREM 17.1 *If a function f is H_G-integrable on $L \in \mathcal{I}_G$ then the indefinite H_G-integral $F(K) = (H_G)\int_K f$ as an additive function on the set of all $\mathcal{B}_G$-subintervals K of L, is $\mathcal{B}_G$-differentiable almost everywhere on L and*

$$D_G F(g) = f(g) \quad \text{a.e. on } L. \tag{17.8}$$

The following theorem related to the problem of recovering the primitive was proved in [8].

THEOREM 17.2 *Let an additive function $F : \mathcal{I}_G \to I\!R$ be $\mathcal{B}_G$-differentiable everywhere on $L \in \mathcal{I}_G$ outside of a set E with $\mu_G(E) = 0$, and $-\infty < \underline{D}_G F(x) < \overline{D}_G F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}_G$-continuous. Then the function*

$$f(x) := \begin{cases} D_G F(x), & \text{if it exists,} \\ 0, & \text{if } x \in E \end{cases}$$

is H_G-integrable on L and F is its indefinite H_G-integral.

To compare H_G-integral with the Lebesgue integral with respect to the Haar measure μ_G we use the known fact (see [1]) that the group G can be mapped on $[0, +\infty]$ by a measure preserving mapping ϕ which is one-one up to a countable set. As the Lebesgue integral is invariant under measure preserving mapping, then a function f defined on a compact subset K of G is Lebesgue integrable on K if and only if the function $f(\phi^{-1})$ is Lebesgue integrable on $\phi(K)$ with the same value of the integral. As we have mentioned in the introduction such mapping is related to introducing of certain ordering in the group G. So we have a class of mappings each of them being measure preserving but the value of the Lebesgue integral of $f(\phi^{-1})$ is the same for all ϕ from this class.

We can use now the known relation between Henstock and Lebesgue integrals on the interval of the real line. The basis $\mathcal{B}_G$ and the H_G-integral with respect to it, after mapping ϕ, are transformed into the so called P-adic basis and the P-adic Henstock integral (H_P-integral, see [6]) on the real line. As P-adic Henstock integral is known to be a generalization of the usual Henstock integral, then it is also a generalization of the Lebesgue integral. So we have

$$(L)\int_K f d\mu_G = (L)\int_{\phi(K)} f(\phi^{-1})d\mu = (H_P)\int_{\phi(K)} f(\phi^{-1}) = (H_G)\int_K f.$$

Therefore going back to the group setting we obtain that H_G-integral on G is a generalization of the Lebesgue integral on G. Hence we get

THEOREM 17.3 *If a function f is summable on $L \in \mathcal{I}_G$ with respect to μ_G then it is H_G-integrable on L and the values of the integrals coincide.*

Moreover as a consequence of Theorem 17.1 and Theorem 17.3 we obtain:

THEOREM 17.4 *If a function f is summable on $L \in \mathcal{I}_G$ with respect to μ_G then the indefinite integral $F(K) = \int_K f d\mu_G$ as an additive function on the set of all $\mathcal{B}_G$-subintervals K of L, is $\mathcal{B}_G$-differentiable almost everywhere on L and*

$$D_G F(g) = f(g) \quad \text{a.e. on } L. \tag{17.9}$$

Another consequence of the Theorem 17.3 combined with Theorem 17.2 is

THEOREM 17.5 *Let an additive function $F : \mathcal{I}_G \to I\!R$ be $\mathcal{B}_G$-differentiable everywhere on $L \in \mathcal{I}_G$ outside of a set E with $\mu_G(E) = 0$, and $-\infty < \underline{D}_G F(x) < \overline{D}_G F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}_G$-continuous. Assume also that derivative $f = D_G F$ is summable on L. Then F is its indefinite Lebesgue integral.*

4. Application to the Series with Respect to the Characters

We consider here the case when the group G is compact and so the chain (17.1) is reduced to the one-side sequence

$$G = G_0 \supset G_1 \supset G_2 ... \supset G_n \supset .... \tag{17.10}$$

In this case the H_G-integral is defined on the whole group G. Moreover the group Γ of characters of the group G is discrete now (see [1]) and it can be represented as a sum of increasing chain of finite subgroups

$$\Gamma_0 \subset \Gamma_{-1} \subset \Gamma_{-2} \subset ... \subset \Gamma_{-n} \subset ... \tag{17.11}$$

where $\Gamma_0 = \{\gamma^{(0)}\}$ with $(g, \gamma^{(0)}) = 1$ for all $g \in G$.

So the characters γ constitute a countable orthogonal system on G with respect to normalized measure μ_G and we can consider a series

$$\sum_{\gamma\in\Gamma} a_\gamma \gamma \tag{17.12}$$

with respect to this system. We define a convergence of this series at a point g as the convergence of its partial sums

$$S_n(g) := \sum_{\gamma\in\Gamma_{-n}} a_\gamma (g,\gamma) \tag{17.13}$$

when n tends to infinity.

We associate with the series (17.12) a function F defined on each coset K_n by

$$F(K_n) := \int_{K_n} S_n(g) d\mu_G. \tag{17.14}$$

This type of function is often referred to as quasi-measure (see for example [11]).

It follows easily from Lemma 17.2 that F is an additive function on the family of all $\mathcal{B}_G$-intervals.

By Lemma 17.1 the sum S_n, defined by (17.13), is constant on each K_n. Then (17.14) implies

$$S_n(g) = \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{17.15}$$

THEOREM 17.6 *The series* (17.12) *is the Fourier series of some integrable function f if and only if the function F associated with this series by expression* (17.14) *coincides on each $\mathcal{B}_G$-interval I with the indefinite integral $\int_I f$.*

Proof. This can be easily proved by the arguments used in [2, Theorem 2.8.1] for the Vilenkin-Price system.

The next two lemmas are immediate consequences of the equality (17.15).

LEMMA 17.3 *If the series* (17.12) *converges at some point $g \in G$ to a value $f(g)$ then the associated function F (see* (17.14)*) is $\mathcal{B}_G$-differentiable at g and $D_G F(g) = f(g)$. Moreover if the series* (17.12) *satisfies at a point g the conditions*

$$-\infty < \liminf_{n\to\infty} \mathrm{Re} S_n(g) \le \limsup_{n\to\infty} \mathrm{Re} S_n(g) < +\infty, \tag{17.16}$$

$$-\infty < \liminf_{n\to\infty} \mathrm{Im} S_n(g) \le \limsup_{n\to\infty} \mathrm{Im} S_n(g) < +\infty, \tag{17.17}$$

then the associated function F satisfies the inequalities

$$-\infty < \underline{D}_G \mathrm{Re} F(g) \le \overline{D}_G \mathrm{Re} F(g) < +\infty, \tag{17.18}$$

$$-\infty < \underline{D}_G \mathrm{Im} F(g) \leq \overline{D}_G \mathrm{Im} F(g) < +\infty. \tag{17.19}$$

LEMMA 17.4 *If the partial sums* (17.13) *satisfy at a point g the condition*

$$S_n(g) = o\left(\frac{1}{\mu_G(K_n(g))}\right) \tag{17.20}$$

then the associated function F is $\mathcal{B}_G$-continuous at the point g.

The next lemma gives a sufficient condition for the assumption (17.20) of the previous lemma to hold.

LEMMA 17.5 *Suppose that the coefficients $\{a_\gamma\}$ of a series* (17.12) *satisfy the condition*

$$\max_{\gamma \in \Gamma_{-(n+1)} \setminus \Gamma_{-n}} |a_\gamma| \to 0 \ \text{ if } \ n \to \infty, \tag{17.21}$$

then (17.20) *holds for partial sums $S_n(g)$ at each point $g \in G$.*

THEOREM 17.7 *Suppose that the partial sums* (17.13) *of the series* (17.12) *converge almost everywhere on G to a summable function f and satisfy the conditions* (17.16) *and* (17.17) *everywhere on G except on a countable set M, where* (17.20) *holds. Then* (17.12) *is the Fourier-Lebesgue series of f.*

Proof. Applying (17.15) we get that for any point g at which the series (17.12) converges to $f(g)$, the function F defined by (17.14) is $\mathcal{B}_G$-differentiable at g with $D_G F(g) = f(g)$.

According to Lemma 17.3, (17.16) and (17.17) imply inequalities (17.18) and (17.19) everywhere on G, except on the set M where by Lemma 17.4 F, together with $\mathrm{Re}F$ and $\mathrm{Im}F$, is $\mathcal{B}_G$-continuous.

Therefore, by Theorem 17.5, $\mathrm{Re}F$ and $\mathrm{Im}F$ are Lebesgue integrals of $\mathrm{Re}f$ and $\mathrm{Im}f$. Hence F is the indefinite integral of f, and using Theorem 17.6 we complete the proof.

REMARK 17.2 *In view of Lemma 17.5 we can replace the condition (17.20) by the condition (17.21) in the assumption of the above theorem.*

Let $f : G \to \mathbf{C}$ be summable on G. Then the partial sums $S_n(f, g)$ of the Fourier series of f with respect to the system of characters can be represented, according to Theorem 17.6 and formula (17.15), as

$$S_n(f, g) = \frac{1}{\mu_G(K_n(g))} \int_{K_n(g)} f.$$

From this equality together with differentiability property of the indefinite Lebesgue integral (see Theorem 17.4) follows

THEOREM 17.8 *The partial sums $S_n(f, g)$ of the Fourier series of a summable function f on G are convergent to f almost everywhere on G.*

5. The Inversion Formula for Transform in the Locally Compact Case

To simplify our notation we shall put in this section $K = K_0$, $[g] := g_K$, $\{g\} := \{g\}_0$, so that representation (17.3) with $n = 0$ for any element g of some coset K of G_0 can be rewritten in the form $g = [g] + \{g\}$ where $[g]$ is a fixed element of K and $\{g\} \in G_0$. Similarly we shall use sometimes the notation $[\gamma] := \gamma_J$ to underline duality, so the representation (17.5) for any element γ of some coset J of Γ_0 can be rewritten in the form $\gamma = [\gamma] \cdot \{\gamma\}$ where $[\gamma]$ is a fixed element of J and $\{\gamma\} \in \Gamma_0$.

Using this notation and the properties of a character γ we can write

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot (\{g\}, \{\gamma\}) \cdot ([g], \{\gamma\}). \tag{17.22}$$

Now we observe that:

1) $\{g\} \in G_0$ and $\{\gamma\} \in \Gamma_0 = G_0^{\perp}$. So $(\{g\}, \{\gamma\}) = 1$ and we can eliminate $(\{g\}, \{\gamma\})$ from representation (17.22) getting

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot ([g], \{\gamma\}). \tag{17.23}$$

2) $[\gamma] \in \Gamma_{-m(\gamma)} = G_{m(\gamma)}^{\perp}$ where $m(\gamma) \geq 0$ and $[\gamma]\lceil_{G_0}$ is a character of the subgroup G_0.

3) $([g], [\gamma])$ is constant if g belongs to a fixed coset of G_0 and γ belongs to a fixed coset of Γ_0.

4) Using the duality between G and Γ we can state that g represents a character of Γ and, similarly to the property 2), $[g]\lceil_{\Gamma_0}$ is a character of Γ_0. So $([g], \{\gamma\})$ is a value of this character at the point $\{\gamma\}$.

Therefore, according to (17.23), if g belongs to a fixed coset of G_0 and γ belongs to a fixed coset of Γ_0, we can represent (g, γ), up to a constant multiplier $([g], [\gamma])$, as a product of $(\{g\}, [\gamma])$ considered as a value of the character $[\gamma]$ at $\{g\}$, and $([g], \{\gamma\})$ considered as a value of the character $[g]$ at $\{\gamma\}$.

Now we obtain a generalization of Theorem 17.7 for a locally compact case.

THEOREM 17.9 *Assume that G is a group described in Section 2, Γ being its dual group. Let $a(\gamma)$ be a locally summable function and*

$$\lim_{n\to\infty} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma)\, d\mu_\Gamma = f(g) \tag{17.24}$$

a.e. on G, where f is a locally summable function on G. Moreover everywhere on G expect a countable set T we have

$$\limsup_{n\to\infty} \left| \int_{\Gamma_{-n}} a(\gamma)(g, \gamma)\, d\mu_\Gamma \right| < +\infty. \tag{17.25}$$

and for $g \in T$ we have

$$\lim_{n\to\infty} \mu(K_n(g)) \int_{\Gamma_n} a(\gamma)(g,\gamma)d\mu = 0 \tag{17.26}$$

Then the function $a(\gamma)$ can be recovered from f by the following inversion formula:

$$a(\gamma) = \lim_{n\to\infty} \int_{G_{-n}} f(g)\overline{(g,\gamma)}d\mu_G \quad a.\,e.\ on\ X. \tag{17.27}$$

Proof. The proof is similar to the one of [6, theorem 9] although we use here a weaker assumption on the convergence of the integral in (17.24) and have in mind the convergence of a series as it is understood in Section 4 (see (17.13)). Having fixed a coset K suppose that $g \in K$ and let J denote any coset of Γ_0. Then by (17.23)

$$\int\limits_{\Gamma_{-n}} a(\gamma)(g,\gamma)d\mu_\Gamma = \sum_{J\subset\Gamma_{-n}} \int_J a(\gamma)(\{g\},[\gamma])\cdot([g],[\gamma])\cdot([g],\{\gamma\})d\mu_\Gamma$$

$$= \sum_{J\subset\Gamma_{-n}} (\{g\},\gamma_J)\cdot \int_J a(\gamma)([g],[\gamma])\cdot([g],\{\gamma\})d\mu_\Gamma. \tag{17.28}$$

The last sum can be considered as a partial sum

$$\sum_{J\subset\Gamma_{-n}} b_J^{(K)}(\{g\},\gamma_J) \tag{17.29}$$

of the series with respect to the system of characters $\{\gamma_J\}_J$, at the point $\{g\}$, with the coefficients

$$b_J^{(K)} = \int_J a(\gamma)([g],[\gamma])(g_K,\{\gamma\})d\mu_\Gamma.$$

According to the assumption (17.24) and the equality (17.28) this series is convergent almost everywhere on K to a function $f(g)$ which by hypothesis is summable on K.

Introducing the variable $t = \{g\} \in G_0$ we can consider this series to be convergent almost everywhere on G_0 to the summable function $p(t) = f(g_K + t)$. The partial sums (17.29) are bounded according to (17.25) and (17.28) except a countable set $M = \{t \in G_0 : g_K + t \in T\}$ where (17.26) holds which corresponds to the condition (17.20) applied to $t \in M$.

Therefore by Theorem 17.7 the coefficients $b_J^{(K)}$ are the Lebesgue-Fourier coefficients of $p(t)$, with respect to characters γ_J, i.e.,

$$b_J^{(K)} = \int_J a(\gamma)([g],[\gamma])(g_K,\{\gamma\})d\mu_\Gamma \tag{17.30}$$

$$= \int_{G_0} p(t)\overline{(\{g\},\gamma_J)}d\mu_G = \int_K f(g)\overline{(\{g\},\gamma_J)}d\mu_G$$

(in the last equality we use the obvious invariance of Lebesgue integral under translation). By observation 3), $([g],[\gamma])$ is constant when $g \in K$ and $\gamma \in J$ with $|([g],[\gamma])| = 1$. Hence (17.30) implies

$$\int_J a(\gamma)(g_K,\{\gamma\})d\mu_\Gamma = \int_K f(g)\overline{([g],[\gamma])(\{g\},\gamma_J)}d\mu_G. \tag{17.31}$$

Now we notice that for each fixed J, the value

$$\int_J a(\gamma)(g_K,\{\gamma\})d\mu_\Gamma$$

is the Fourier coefficient, with respect to the character $\overline{g_K}$, of the summable function $a(\gamma) = a([\gamma]+\{\gamma\})$ considered as a function of $\{\gamma\} \in \Gamma_0$. Applying Theorem 17.8 to this Fourier series, we get

$$\lim_{n\to\infty} \sum_{K\subset G_{-n}} \int_J a(\gamma)(g_K,\{\gamma\})d\mu_\Gamma \cdot \overline{(g_K,\{\gamma\})} = a([\gamma]+\{\gamma\}) = a(\gamma)$$

for almost all values of $\{\gamma\}$ on Γ_0, i.e., a.e. on J. Hence using (17.31) and (17.23) we compute

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{K\subset G_{-n}} \int_J a(\gamma)(g_K,\{\gamma\})d\mu_\Gamma \cdot \overline{(g_K,\{\gamma\})} \\
&= \lim_{n\to\infty} \sum_{K\subset G_{-n}} \int_K f(g)\overline{([g],[\gamma])(\{g\},\gamma_J)}d\mu_G \cdot \overline{(g_K,\{\gamma\})} \\
&= \lim_{n\to\infty} \int_{G_{-n}} f(g)\overline{(\{g\},\gamma_J)\cdot(g_K,\{\gamma\})\cdot([g],[\gamma])}d\mu_G \\
&= \lim_{n\to\infty} \int_{G_{-n}} f(g)\overline{(g,\gamma)}d\mu_G = a(\gamma) \quad \text{ a.e. on } J.
\end{aligned}$$

The last equality is true for any J, so we get (17.27), completing the proof.

We remark that the Theorems 17.7 and 17.9 can be generalized by the same methods to the case of any integral which is compatible with the H_G-integral (for example the analogue of the Denjoy-Perron integral on the group), but it is not true even for the Vilenkin-Price system if we use here the Denjoy-Khintchine integral (see [5]).

This kind of theorem becomes true for Denjoy-Khintchine integral if we put some additional hypothesis on the group and on the type of convergence (see [10]).

References

[1] Agaev, G.N., Vilenkin, N.Ya., Dzhafarli, G.M., Rubistein, A.I., *Multiplicative System of Functions and Harmonic Analysis on Zero-dimensional Groups*, Baku 1981, (in Russian).

[2] Golubov, B., Efimov, A., Skvortsov, V., *Walsh Series and Transforms: Theory and Applications*, Kluwer Academic Publishers, 1991.

[3] Ostaszewski, K.M., "Henstock integration in the plane", *Memoirs of the AMS*, Providence, Vol. **63**, No. 353, 1986.

[4] Skvortsov, V.A., "$\mathcal{P}$-adic Henstock integral in inversion formula for multiplicative transform", *Real Analysis Exchange, 27th Summer Symposium Conference Reports*, June 2003, 93-97.

[5] Skvortsov V.A., and Koroleva M.P., "Series in multiplicative systems convergent to Denjoy-integrable functions", *Mat. Sb.*, (1995) **186**, No. 12, 129-150, (Engl. transl. *Math. USSR Sb.*, (1995) **186**, No. 12, 1821-1842.

[6] Skvortsov, V.A., Tulone, F., "Henstock type integral in harmonic analysis on zero-dimensional groups", *J. Math. Anal. Appl.*, **322**, (2006), 621-628.

[7] Skvortsov, V.A., Tulone, F., "$\mathcal{P}$-integral in the theory of series with respect to characters of zero-dimensional groups", *Vestnik Moskov. Gos. Univ. Ser. Mat. Mekh.*, **1**, (2006), 25-29, Engl. transl. *Moscow Univ. Math. Bull.*, **61**, (2006), 27-31.

[8] Skvortsov, V.A., Tulone, F., "Kurzweil-Henstock type integral on zero-dimensional group and some of its application", *Czech. Math. J.*, (in press).

[9] Thomson, B.S., "Derivation bases on the real line", *Real Anal. Exchange*, **8**, (1982/83), 67-207 and 278-442.

[10] Tulone, F., "Denjoy and $\mathcal{P}$-path integrals on compact groups in an inversion formula for multiplicative transforms", *Tatra Mountain Mathematical Publications*, (in press).

[11] Wade, W.R., Yoneda, K., "Uniqueness and quasi-measures on the group of integers of a p-series field", *Proc. Amer. Math. Soc.*, **84**, (1982), 202-206.

[12] Zygmund, A., *Trigonometric Series*, Cambridge University Press, Vol. 2, 1955.