Chapter 12

GIBBS DERIVATIVES 40 YEARS AFTER THE INTRODUCTION OF THE CONCEPT

Notion, extensions, and generalizations - a brief overview

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Abstract In this paper we present a short account of the development of the theory of Gibbs differentiation on the occasion of 40th anniversary of its introduction [20], and the 35th of Butzer-Wagner dyadic derivative [10] as well as the 30th anniversary of Onneweer's extension of the theory to Vilenkin groups [40], the two most important results in two different ways of generalization of the Gibbs' introductory result. We also attempt to give a rather general characterization of Gibbs derivatives viewed as a class of differential operators on different not necessarily Abelian groups.

1. Background and motivation

Signals are physical conveyors of information. They are physical processes which spread in space-time and, therefore, are conveniently modeled by elements of some functional spaces. In this setting, signals are frequently identified with their mathematical models.

The following classes of signals can be distinguished in signal theory [45].

1 Signals modeled by real variable functions are called *continuous signals*. The continuous signals of the continuous amplitude are *analog* or *analogue signals*.

2 *Discrete signals* are modeled by discrete functions, i.e., they are usually considered as functions on the set of integers Z or on some of its subsets Z_q of integers less than some given q.

3 Discrete signals taking their values in some finite sets are *digital signals*.

Irrespective to which class belong, signals should be processed in order to extract, interpret, and exploit the information contained in them. Mathematics provide a theoretical base for disclosing various signal processing tools. The *Fourier analysis* and *differential calculus* are certainly the two most powerful among them.

The Fourier analysis expresses two important principles, the *superposition* and the *linearity*, both principles often present in engineering considerations of physical phenomena. In this way, the Fourier analysis provides means for mathematical description of a system in an approach frequently used in engineering practice, the decomposition of a complex problem into finitely or countably many simpler subproblems.

In the same ground, differential operators are conveniently used to express the direction as well as the rate of change of a signal at the input and/or output of a system, providing in this way the information about the state of the system considered.

A conviction prevalent for a long time was that signals existing in reality could be described adequately exclusively by functions on the real line R. A reason for that could be the conjecture that the topology of space-time is well represented by the topology of the real line. For that reason the Fourier analysis and differential calculus were first established, and for many years restricted, almost ultimately to R, the continuum which is one of the most sophisticated structures in mathematics.

This fact has been noticed as a *paradox of history* [25]. The mentioned conviction was greatly changed by the recognition of the so-called sampling theorem [36], [56], which states that, under certain conditions, continuous signals could be adequately represented by their discrete counterparts.

These conditions are conveniently expressed in terms of Fourier coefficients, and we can realize again the importance of Fourier analysis in signal processing.

It could be stated that the interest in practical engineering applications of discrete structures and functions defined on them sprang after the publication of famous Shannon's paper [56], although the essence of sampling theorem were known much earlier, see for example, [31].

In this settings the practical applicability of Fourier analysis were further supported by the rediscovery of the *Fast Fourier transform*, FFT, [16] which is a fast algorithm for the calculation of Fourier coefficients on finite groups. We use the term rediscovered since, similarly as in the case of sampling theorem, the essence of FFT was known to some authors of earlier times, as it is well documented in [30]. Today, FFT should be appreciated as a corner stone of the theory of fast signal processing transformations and the key part of many signal processing algorithms.

Among discrete functions, the *switching functions* defined as the mapping $f : \{0,1\}^n \rightarrow \{0,1\}$, $n \in N$ - the set of natural numbers, are probably most widely used, since all digital devices at the hardware level are realized by circuitry based upon two stable state basic elements.

Actually, the *switching theory* dealing with switching functions, provides a mathematical base for the description of behavior and functioning of digital devices and the representation of signals by binary sequences.

The *discrete Walsh transform* [73] is the basis for the Fourier analysis compatible with switching functions, the *Walsh-Fourier analysis*, since these functions can be conveniently viewed as a particular subset of functions on *finite dyadic group*, which consists of the set of 2^n binary n-tuples $x = (x_1, \ldots, x_n)$, $x_i \in \{0,1\}$, under the componentwise addition modulo 2.

Recall that the *Walsh functions*, the basis for the Walsh-Fourier analysis, are the group characters of the dyadic group [17], and therefore the Fourier analysis on that group is based upon them in the same way as the classical Fourier analysis is based upon the exponential functions e^{jwx}, the group characters of the real line R viewed as a particular locally compact Abelian group [53]. It is the same for the discrete Walsh functions viewed as the group characters of finite dyadic groups [1], [2].

The Walsh functions take only two different values ± 1 and, therefore are also in that respect compatible with binary-valued switching functions. This fact ensures at the same time the simplicity of computation with Walsh functions. As we noted, the Walsh transform is a particular case of Fourier transform on groups, and therefore, can be performed by fast transform algorithms derived as a particular case of FFT, see for example [1], [3], [54]. The fast Walsh transform can be computed without multiplication, which make it the computationally most efficient among Fourier transforms on different groups, since the multiplication is usually a more time consuming operation than addition when realized with present software and hardware technological resources.

In this way, the group theoretic approach to Fourier analysis, which suggests to use group characters for locally Abelian and unitary irreducible group representations for compact non-Abelan groups as kernels for the Fourier transform, enables a unique way of the extension of this theory to structures other than R and in particular to discrete structures.

Regarding differentiation, the extension was not so straightforward. The discrete and piecewise constant functions used with digital devices cannot be differentiated in the *Newton-Leibniz* sense and the need for an appropriate differential operator was apparent from almost the beginning of the use of discrete functions in engineering practice, especially in communications [39], [50].

It is quite understandable that first results were set in switching theory by the introduction of the *Boolean difference* [39], [50], since at the hardware level, digital communications are implemented through binary valued sequences.

The theory of this operator was established in [2],[69] and its application proved very useful in many areas of logic design and digital communications.

The term *Boolean differential calculus* is now often used, see for example explanation given in [70], although the Boolean difference can hardly be accepted as a proper differentiator, since it does not permit to distinguish the change of a switching variable from 0 to 1 from that of 1 to 0.

In any way, the Boolean difference is acting on the set of switching functions and, therefore, is a very particular answer to the problem of differentiation of discrete and piecewise constant functions used with digital devices.

The interest in Walsh functions, which raised in latest sixties, provide another motive for investigations on differentiation of such functions. At that time it was apparent a desire to consider the Walsh functions as a particular case of *special functions* [4], examples of which are Bessel, Chebyshev, Laguerre, Hermite, Lagrange, Legendre, etc., [4].

Special functions are usually generated as the solutions of some generating differential equations and, therefore, a corresponding differential operator was needed.

In the pioneering work in 1967, J. Edmund Gibbs proposed the following definition by attempting to answer to this desire for a differential operator in the case of discrete Walsh functions.

DEFINITION 12.1 *(Gibbs finite derivative)*
For a function f defined on finite dyadic group of order 2^n, the finite dyadic derivative $f^{(1)}$ is defined as

$$f^{(1)}(x) = -2^{-1}\sum_{r=0}^{n-1}(f(x \oplus 2^r) - f(x))2^r, \quad \forall x \in \{0, \ldots, 2^n - 1\}.$$

The finite dyadic derivative of a function f we also denote by Df. This operator D, also called *logical derivative* [23], [24], has the discrete Walsh functions as its eigenfunctions.

It follows that the discrete Walsh functions emerge as the solutions of the first order dyadic differential equation

$$Df - \lambda f = 0, \quad \lambda \in \{0, 1, \ldots, 2^n - 1\}.$$

As it is explained in [21], the operator was introduced quite independently on any work on Boolean difference. However, by relating these two operators [18], [19], the application of the finite dyadic derivative in the same areas where Boolean difference was already applied proved very interesting [32]. At the same time, the interpretation of finite dyadic derivative as the linear combination of partial dyadic derivatives, similar to Boolean differences relative to particular coordinates [64], offered a mean for the derivation of fast algorithms for the calculation of Gibbs derivatives on finite groups, the later generalizations of finite dyadic derivative [66]. Note that the term Gibbs derivatives was established and confirmed in particular at the *First international Workshop on Gibbs derivatives* in 1989 [9] in order to denote a broad family of differential operators representing the generalizations and extensions or were inspired by the finite dyadic derivative originated by J. Edmund Gibbs [21].

2. Generalizations of the finite dyadic Gibbs derivative

The great interest in practical engineering applications of Walsh analysis which sprang after the publication of Harmuth's paper in 1960 [29] provided a very suitable environment for further work on Gibbs derivatives. The activity in the area of Walsh analysis is best illustrated by the fact that from 1970 to 1974 the specialized conferences completely devoted to Walsh and related functions and their applications were organized in U.S.A. Moreover, in 1971, 1973, and 1975 the conferences on this particular subject were organized also in England, so that two conferences per year on the same subject were organized. This research work intended towards applications in computer science and engineering, leaded to the extension of the theory of Walsh and related functions into the so-called *spectral techniques* [5], [33], [34], [38], which are from that time the subject of specialized workshops or are discussed at standard sections at many conferences and meetings on signal processing and multiple-valued logic. Research papers on these subjects are accepted by many mathematical and engineering journals. For example, the journal *IEEE Transactions on Electromagnetic Compatibility* published by IEEE Press had the Associate Editors for Walsh and non-sinusoidal functions. This position was served very successfully by Henning F. Harmuth for many years.

The great activity in the area of Walsh functions have resulted among other things in some interesting generalizations and extensions of finite dyadic derivative. It should be noticed that 40 papers on the subject were presented at the conferences on Walsh and related functions from 1970 to 1975. Today the bibliography on Gibbs derivatives consists of over 277 items published by 69 authors from 14 nations all over the World. Among these, the probably most important and certainly most widely discussed result is the *Butzer-Wagner dyadic derivative* introduced 35 years ago [10].

Roughly speaking, two different ways of generalization of finite dyadic derivative can be distinguished. In essence, they are based upon two alternative interpretations of the basic property of finite dyadic derivative that the discrete Walsh functions are the eigenfunctions of that operator.

First, this property implies that the Walsh functions are infinitably many times dyadicaly differentiable. Various extensions of the finite dyadic Gibbs derivative were aimed at extending the class of differentiable functions.

The second implication concerns to the relationship of finite dyadic derivative with Walsh transform similar to the relationship of the Newton-Leibniz derivative with the Fourier transform in classical analysis on R. This was a basis for generalizations of Gibbs differentiation to groups other than the dyadic group.

Extension of the class of differentiable functions

The way of generalizations based upon the first implication, that started 35 years ago by Butzer and Wagner [10], is devoted to the extension of the class of functions differentiable in some sense, in this case, the dyadic sense. The approach were originated in Walsh-Fourier analysis on $[0, 1]$ and, therefore, mainly concerns functions on that interval. Recall that this interval can be identified with the infinite dyadic group consisting of countably many copies of the finite dyadic group of order 2 enriched with the product topology owing to the mapping

$$\lambda(x_1, x_2, \ldots) = \sum_{i=1}^{\infty} x_i 2^{-i}, \quad x_i \in \{0, 1\}.$$

The Walsh functions, being the characters of the dyadic group [17], form a complete orthonormal system in the space L^2 of measurable functions square integrable on the interval $[0, 1]$.

Butzer and Wagner [10] extended the concept of dyadic differentiation from finite dyadic group to the infinite dyadic group, or alternatively interval $[0, 1]$, by introducing a derivative D on $[0, 1]$ which eigenfunctions are the Walsh functions in the *Kaczmarcz ordering*, i.e., for which

$$D(wal_k) = k \cdot wal_k, \quad k = 0, 1, \ldots, \tag{12.1}$$

where wal_k denotes the Walsh function of order k in the Kaczmarz ordering.

Butzer and Wagner also defined a derivative which expresses the same relation with respect to the *Paley ordered* Walsh functions [11].

Denoting by X either the space ϕ of functions continuous on $[0,1]$ or one of the spaces L^p, $1 < p < \infty$ of measurable functions whose p-th power is integrable over the interval $[0,1]$, the Butzer-Wagner derivative for *Paley ordered Walsh functions* can be described as follows.

DEFINITION 12.2 *(Butzer-Wagner derivative for Paley ordering)*
For a function $f \in X$ for which the sequence of functions

$$d_n(f,x) = \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x \oplus 2^{-j-1})),$$

converges in the norm of X the strong Butzer-Wagner dyadic derivative *is defined as the limit*

$$\lim_{n\to\infty} d_n(f,x).$$

Notice that Butzer and Wagner [13] further introduced the concept of the *pointwise dyadic derivative* by saying that a function from $[0,1]$ has the pointwise dyadic derivative at a point $x \in [0,1]$ if the sequence of real numbers $\{d_n(f,x)\}$ converges as $n \to \infty$.

The relation (12.1) is true for all $x \in [0,1]$ also for the pointwise Butzer-Wagner dyadic derivative with respect to the generalized Walsh functions $\Psi_y(x)$, i.e.,

$$D(\Psi_y)(x) = |y|\Psi_y(x),$$

for x, y taking values in the dyadic field.

As it is noticed in [6], [71], the dyadic derivative was especially adopted to functions having many jumps and possessing just a few and also short intervals of constancy. Even functions having a denumerable set of discontinuities like the well-known *Dirichlet function* can be dyadic differentiated on $[0,1]$ [76].

The extended dyadic derivative [7], [8] based upon the works by Butzer and Wagner [10], [11], [13] and He Zelin [77] is applicable also to piecewise polynomial functions, i.e., to functions which are made up entirely of polynomial pieces between the consecutive jumping points.

Extensions to functions on different groups

A possible characterization of Butzer-Wagner dyadic derivative can be given in terms of Walsh series coefficients $S_f(w)$ of f as

$$S_{Df}(w) = wS_f(w), \quad w = 1, 2, \ldots, \tag{12.2}$$

which is a consequence of the fact that Walsh functions are the eigenfunctions of this differential operator. Obviously, the same is true for dyadic derivative on finite groups [25].

The second way of generalization of the finite dyadic derivatives devoted to the transfer of the concept of differentiation to structures other than real line R is based just upon that characterization of finite dyadic derivative and the Butzer-Wagner dyadic derivatives.

The basic idea is relatively simple. The Newton-Leibniz derivative can be viewed as the linear operator mapping the exponential functions e^{jwx}, the characters of the real line R, into jw times of themselves.

The similar holds for the dyadic derivatives, relation (12.1), where the real line is replaced by the dyadic group and the exponential functions by the Walsh functions, the characters of the dyadic group. This relation reads as the relation (12.2) in the transform domain owing to the orthogonality of Fourier transform on groups. The theory should be extended to other locally compact Abelian or compact non-Abelian groups by the replacement of Walsh functions by the characters of the corresponding Abelian group or by unitary irreducible representations of the non-Abelian groups.

It is interesting to note that from this group-theoretic approach point of view there is apparent some parallelism between the ways of development of abstract harmonic analysis and the theory of Gibbs differential calculus on groups.

Recall that the abstract harmonic analysis is the mathematical discipline developed from the classical Fourier analysis by the replacement of the real line R, which is a particular locally compact Abelian group, by arbitrary locally compact Abelian or compact non-Abelian group. The same has been done in the case of Gibbs derivatives by using the relation (12.2) as a defining relationship of these operators. The first attempts in this direction were given again by J.E. Gibbs and his associates [14], [25].

A considerable extension of Gibbs differentiation on groups were given 30 years ago by Cornelis W.Onneweer [40] who introduced a Vilenkin group analogue of the dyadic derivative showing that the characters of the Vilenkin group are the eigenfunctions of the introduced differential operator and argued that the Butzer-Wagner characterization (12.2) caries over with extra work to this setting. See, also [49], [71].

In the similar way, there have been defined L^r-weak p-adic derivative, the adjacent p-adic derivative, the partial p-adic derivative [51], [74], [78]. See also [75].

Several other authors consider this way of generalizations of Gibbs differential calculus to other structures including also the discrete structures.

For example, Pál [46] defined the dyadic derivative Df on the dyadic field, i.e., for functions $f \in L^1(0, \infty)$ and showed that the Walsh transform F defined in [17], interacts with D as follows: if Df exists then $F(Df)(y) = yF(y)$ and $D(Fy)(x) = F(xf(x))$ if $xf(x) \in L^1(0, \infty)$.

In [48], Pál constructed an indefinite integral for D and proved a fundamental theorem of calculus in this setting. The extension of the dyadic differentiation to R+ were also considered, see for example [12], [47], as well as to local fields [44].

Recently, Golubov [27] introduced the modified strong dyadic integral J_α and the fractional derivative $D^{(\alpha)}$ of order $\alpha > 0$ for functions from the Lebesgue space $L(R_+)$. Established are criteria for existence of these integrals and derivatives for a given function $f \in L(R_+)$ and determined a countable set of eigenfunctions of these operators.

For the fractional dyadic derivative and integral, proven are in [28], the theorem on differentiation of the indefinite Lebesgue integral of an integrable function at its Lebesgue points, and the theorem on reconstruction of an absolutely continuous function by means of its derivative. These theorems can be viewed as analogues of the theorems of Lebesgue in classical analysis.

A class of generalizations of Gibbs differential calculus concerning both functions defined on the interval $[0, 1]$ and on different discrete Abelian groups is obtained through the replacement of group characters by some other orthogonal systems, as for example, the system of Haar functions [58], discrete Haar functions [67] and generalized Haar functions [68], by an arbitrary orthogonal system [62], or even by an arbitrary bi-orthogonal system [59].

The transfer of the notion of Gibbs differentiation to finite non-Abelian groups was done in [60] and further considered in [61], [63], [65]. An approach to the extension of definition of Gibbs derivative to infinite non-Abelian groups was suggested in [61] following the idea of Butzer-Wagner definition of strong dyadic derivative.

3. Towards a general characterization of Gibbs derivatives

In this section we will attempt to give a characterization of Gibbs derivatives through a group-theoretic approach to the subject.

In order to cover in a uniform way the case of functions on Abelain and non-Abelian groups, we restrict the considerations to the space $K(G)$ of functions defined on a locally compact Abelian or a finite non-Abelian group G taking the values in a field K admitting the existence of a Fourier transform.

In a general ground, the Gibbs derivative of order k of a function $f \in K(G)$, which we denote by $D^k f$ is considered as the linear operator in $K(G)$ satisfying the relationship

$$(F(D^k f))(w) = \phi(w, k)(F(f))(w), \tag{12.3}$$

where F denotes the Fourier transform operator in $K(G)$.

In the most examples $\phi(w,k) = w^k$, but in some cases a scaling factor should be added, see, for example [23], while in a few particular cases the function ϕ differs and is related to the order of group G. For example, in the case of extended Butzer-Wagner dyadic derivative [7], $\phi(w,k) = (w^*(w))^k$, where

$$w^* = \sum_{i=0}^{\infty}(-1)^i w_i 2^i,$$

w_i, being the coefficients in the dyadic expansion of $w \in P$.

It is important to notice that in any case the definition of the function ϕ, and in this way of the Gibbs derivative, relates to the ordering of group characters or unitary irreducible group representations of G.

For example, as noticed above, the Butzer-Wagner dyadic derivative has been defined for the Kaczmarcz and Paley ordered Walsh functions.

From the very beginning of the theory of Gibbs differentiation this was considered as a deficiency of the theory. The problem has been discussed by J.E. Gibbs and several other authors, in particular in details by C.W. Onneweer [41], who endeavored to erase it by suggesting new definitions of derivatives on p-adic and p-series fields.

Another definition of dyadic derivative were offered in [42] and compared with some other definitions of that operator. At the same time, as is noticed in [37], different orderings of group characters or unitary irreducible representations could offer for a given group G the family of Gibbs derivatives some of which could be potentially more convenient than others regarding some concrete applications and numerical calculations. The best ordering of group characters or unitary irreducible representations regarding the efficiency of numerical calculation of Gibbs derivatives on finite groups is determined in [66] using the corresponding results for the implementations of FFT on finite groups. Note that depending on the range of the exponent k the given characterization of Gibbs derivatives extends under appropriate conditions to the fractional Gibbs derivatives, and for $k < 0$ subsumes the concept of Gibbs anti-derivatives. See [11], [55], [63], [65], [76], [77] for some particular examples. The uniqueness of the considered class of differential operators is assured by the requirement that the eigenfunctions of Gibbs derivative are the group characters for Abelian groups and the elements of unitary irreducible representations for finite Abelian groups, i.e.,

$$D^k(\chi_w) = a(w,k)\chi_w, \tag{12.4}$$

for Abelian groups, and

$$D^k R_w^{(i,j)} = a(w,k) R_w^{(i,j)}, \tag{12.5}$$

for non-Abelian groups, where χ_w is the w-th group character of an Abelian, and $R_w^{(ij,)}$ is (i,j)-th element of w-th unitary irreducible representation R_w of a non-Abelian group. Note that the eigenvalues $a(w,k)$ of Gibbs derivatives depend on the ordering of group characters or unitary irreducible group representations in the same way as the function $\phi(w,k)$ depend on that ordering.

If a Gibbs derivative is defined so that $\phi(w,k) = w^k$ in the transform domain, then owing to the orthogonality of Fourier transform, $a(w,k) = w^k$ in the original domain, where w is the index of w-th group character or unitary irreducible representation in the corresponding ordering. For example, w could be the index of w-th Walsh function in Kaczmarcz ordering if the definition of Butzer-Wagner dyadic derivative given in [10] is used, or the index of w-th Walsh function in Paley ordering in the case of Butzer-Wagner dyadic derivative introduced in [11].

Alternative definitions of Gibbs derivatives could yield different eigenvalues. For example, the dyadic derivative derived for $p = 2$ from the Onneweer's definition of dyadic derivative on p-series fields yields different eigenvalues from those of the strong Butzer-Wagner dyadic derivative, since in that case

$$D(wal_k) = 2^n wal_k \quad 2^n \leq k \leq 2^{n+1}, \quad n = 0, 1, \ldots. \tag{12.6}$$

Properties of Gibbs derivatives

Besides linearity, the relation (12.3) and its consequence (12.4) or respectively (12.5), the main properties of Gibbs derivatives could be given as follows.

1 *The derivative of a constant*
$Df = 0 \in K$, iff f is a constant function,

2 *Convolution property*

$$D(f_1 * f_2) = (Df_1) * f_2 = f_1 * (Df_2) \in K(G),$$

where $*$ denotes the convolution in $K(G)$.

3 The group characters χ_w for Abelian groups and the functions $f_{i,j}(z) = R^{(i,j)}(x)$ for finite non-Abelian groups are infinitely many times Gibbs differentiable functions.

4 The Gibbs derivatives do not obey the product rule, i.e., it is false that

$$D(f_1 \cdot f_2) = f_1(Df_2) + (Df_1)f_2, \quad \forall f_1, f_2 \in K(G).$$

Notice that the product rule is used as a base for the introduction of *Ritt-Kolchin derivatives* [35], [52] and, therefore, it follows that the Gibbs derivatives can not be involved in that class of differential operators [15].

5 *Shift invariance*

$$D(T_a f) = T_a(Df), \quad \forall f \in K(G),$$

where T_a denotes the shift operator on G defined as $T_a f(x) = f(x \circ a)$, where $\circ$ denotes the group operation on G.

6 *Haar integral of the derivative*

$$\int_G Df = 0 \in K.$$

7 D is a closed operator in $K(G)$.

We infer by the inspection of many particular Gibbs derivatives that the presented general characterization can be given, but we do not have any pretention to subsume all existing particular cases. In any case, the properties 1-7 can be recognized in the presented or in some slightly modified form in the very most of the particular examples of Gibbs derivatives.

For some generalized product rules for finite dyadic derivative, see [64].

The product rule for extended Butzer-Wagner dyadic derivative valid for Walsh functions is given in [8]. Regarding relationship of Gibbs derivatives with some other differential operators, note that a relationship between strong dyadic derivative and classical *Dini derivatives* were given in [13], [57].

The relationship of Gibbs differentiation with classical Newton-Leibniz differentiation were discussed in [22].

4. Closing Remarks

Trying to estimate and appreciate the role of Gibbs derivatives on the occasion of 40th anniversary of its introduction and 35th and 30th anniversary of two important generalizations by P.L. Butzer and H.J. Wagner, C.W. Onneweer, and other authors, we want to point out the following.

1 Gibbs derivatives enable the transfer of differentiation from the real line to different discrete, and otherwise, not necessarily Abelian structures.

2 Through some particular Gibbs derivatives the class of functions differentiable in some sense, in this case, Gibbs sense, is greatly extended.

3 Some Gibbs derivatives have found interesting applications in different areas as, for example, logic design, statistics, sampling theory, system theory and signal processing, see [26] for the relevant references.

4 Efficiently characterized by Fourier coefficients on groups, Gibbs derivatives can be considered as a part of abstract harmonic analysis, giving to

this mathematical discipline another quality, since relate it with a differential calculus in the same way as the classical Fourier analysis is related to Newton-Leibniz differentiation.

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