WALSH AND DYADIC ANALYSIS

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October 18 - 19, 2008
Niš, Serbia

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Dedicated to the memory of
James Edmund Gibbs, who
introduced the notion of
derivation in Walsh and
Dyadic Analysis

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Preface

The *Workshop on Walsh and Dyadic Analysis* has been held on October 18-19, 2007, at the Department of Computer Science, Faculty of Electronics, University of Niš, Serbia.

The Workshop has been dedicated to the memory of James Edmund Gibbs.

This book contains an unpublished paper by J. Edmund Gibbs written in 2004, and elaborated and extended versions of 16 papers presented at the Workshop.

Acknowledgments

Thanks are due to the staff of the *Computer Intelligence and Information Technologies (CIIT) Lab* of the Faculty of Electronics, and members of the Professional Student Clubs, *GNU at the University* and *3D Archeology* for the help in organization and running the Workshop.

Foreword

1. James Edmund Gibbs 1928-2007

Edmund Gibbs was born and brought up in Outer London, the only child of devoted parents. His secondary education was at the local boys' grammar school where he excelled in every academic subject. He organised a school scientific society, and he designed and made a mounting for an astronomical telescope. His school reports were not all praise, however: he was continually in trouble for playing silly pranks thereby disrupting the work of the other pupils. Considered, by his mechanics master, to have exceptional ability, Edmund took up scholarships to Imperial College, London, to read Physics. At the same time, as a by-product, he also gained a first class degree in Mathematics.

Edmund stayed at Imperial College to do a PhD in stellar photometry and, after a brief period working in Canada, did more work in that field at Edinburgh Royal Observatory, as a Cormack Research Fellow of the Royal Society of Edinburgh. From there he was recruited by the Light Division of the National Physical Laboratory (NPL) to collaborate in work on realisation of the primary standard of luminance. Thus, in 1956, he commenced a 32-year career at NPL.

His next project was work on colour-temperature standards. Meanwhile, he gained a reputation for being impeccably dressed and well-groomed, and for driving his small sports car with some dash.

In 1961, he was poached from the Light Division by one Dr Alastair Gebbie to participate as theoretician to a group in the Basic Physics Division working on infra-red Fourier-transform spectroscopy. Dr Gebbie had the reputation of being a very hard taskmaster. Edmund wrote some forty brief working papers in this field, partly resulting from brainstorming sessions with Dr Gebbie and other members of the group.

Initially Edmund wrote Fourier transform programs for the old ACE computer which he then replaced with various Elliott mini computers to provide a regular local service of numerical Fourier transformation for converting interferograms, recorded on paper tape by the group, into spectra. He was one of the first in NPL to bring measurements and the computer closer together.

In 1962, he collaborated in developing the initial idea of dispersive Fourier-transform spectroscopy. A year later, at Dr Gebbie's request for better spectra, he developed, by conflating the techniques of Taylor and Fourier, an asymptotic sequence of sophisticated apodising functions for precise numerical Fourier transformation. His interest in the mathematical techniques deepened as he produced more sophisticated algorithms.

Again at Dr Gebbie's suggestion, in 1966 he investigated the possibility of applying Walsh functions in place of sine and cosine functions (to which they are analogous, taking only the values 1 and -1) in transform spectroscopy. The idea was flawed by the fact that the Michelson interferometer is modelled by a linear time-invariant system, not a linear dyadic-invariant system, to which Walsh functions would be appropriate.

On January 13, 1967, Edmund recognised that Walsh functions could be regarded, with an extended concept of derivative, as eigenfunctions of a differentiator. The new concept of differentiation was generalised and elucidated in collaboration with two colleagues from the University of Bath. Over the next twelve years a sequence of Sandwich Course Students came from that University. This work eventually gave rise to the new mathematical discipline called dyadic analysis, and to a very general concept of differentiation on groups.

Around the time, in 1968, that Dr Gebbie left NPL, Edmund made a second discovery in pure mathematics, that of Fourier analysis in the space of functions from the integers to the Galois field $GF(2)$, these functions being regarded as elements of the dyadic field. This new discipline, extending abstract harmonic analysis in a significant way, was expounded in lectures in 1976 and 1982, but not published.

In 1970, Edmund became involved in the international Walsh-function community, and began to assist in the preparation and examination of PhD students in the field. He began to attract students of Walsh functions and generalised differentiation to work at NPL.

In 1974, he showed that Newton-Leibniz differentiation can be regarded as a special case of Gibbs differentiation. He extended the concepts of frequency and local frequency to functions defined on certain groups in much the same way as the concept of differentiation had previously been extended. In 1976 he collaborated in extending Gibbs differentiation to functions from groups to Galois fields.

In 1977 he published a hardware implementation of an instant Fourier transform in the dyadic field, requiring almost no computation in comparison with the conventional Fast Fourier Transform.

About this time, Edmund was transferred to the Time and Frequency Section. He completed a hypothetico-deductive theory of time presupposing only ordinary logic and set theory. The axiom basis comprises six non-metrical axioms that assign an appropriate set-theoretical structure to time, and two

Figure 0.1. James Edmund Gibbs at about 1970 (Courtesy of Ms. Merion Gibbs).

metrical axioms that assert the existence and uniqueness of the class of periodic time-scales (the uniform time scales desired in practice). Within this framework it was possible to give a definition of time.

He extended the well-known Boolean differential calculus to functions from the dyadic field to $GF(2)$, for which Taylor series have an uncountable number of terms.

In 1982, he discovered a summability convention, consistent with Fourier analysis in the dyadic field, for divergent series of terms in $GF(2)$ (zeros and ones modulo 2).

In his final years at NPL, Edmund concentrated on the study and implementation of algorithms for generating a uniform time-scale from data consisting of intercomparisons, at intervals between the indications of a set of good, but imperfect, clocks.

Edmund's NPL career was somewhat blighted by exasperated (though sympathetic) Divisional Superintendents complaining that he would persist in following his own ideas instead of the official lines! These complaints increased over the years as the activities of NPL inexorably changed from research to the commercialised selling of services.

It is clear that Edmund was considered by all to be a pleasant and friendly colleague, with a quiet sense of humour, very ready to offer advice, to help solve problems, and to praise and encourage the work of others. His Divisional Reports were regarded as erudite and stylistically accomplished (though making difficult reading for the average experimental physicist).

In his retirement, Edmund tried to find as much time as possible for mathematics, begrudging the hours spent in hospital waiting rooms. He happily corresponded with Bryan Ireland, one of the old colleagues from Bath University, who kindly read and commented on many pages of mathematics - including the short paper that has reached the Proceedings of this Workshop.

Since his death early this year, his post mortem has yielded some interesting research material in the field of medicine.

These notes were compiled by Marion Gibbs

2. Edmund Gibbs by Marion Gibbs

Edmund was a kind, calm and interesting person. He was my friend for 30 years and my husband for 23. We were a late marriage. Always thinking rationally, Edmund never acted in haste-hence the difference of 30 and 23!

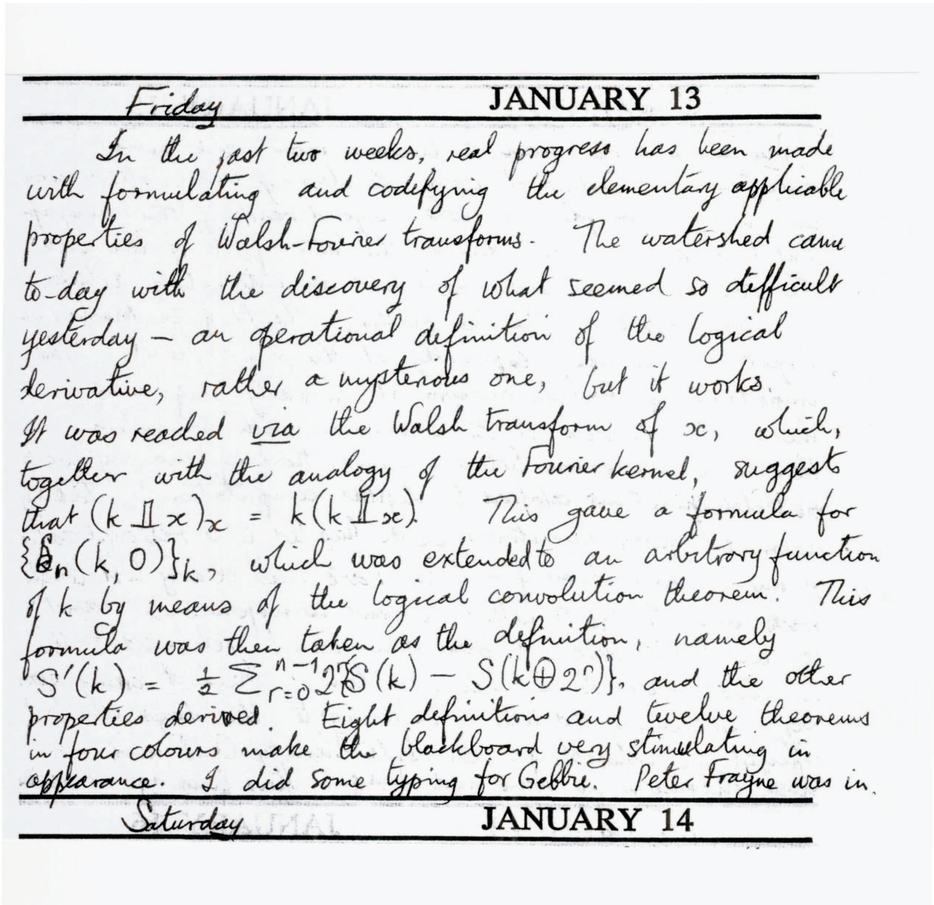
His father worked for the London County Council as an ordinary clerk, but he wrote beautiful handwriting, using impeccable grammar, studied Spanish, and played the violin; and he remained alert and active until his death aged 98. He encouraged Edmund from an early age to write well, read a lot and appreciate good literature. Unlike his son, he was short-tempered and quick to take offence. In later years when father and son met, they quarrelled, but in between meetings they exchanged polite, well-written letters. Edmund's parents were churchgoers and Edmund dutifully went with them, but he insisted to me that he disapproved from the age of three! When Edmund was 26 his mother died of cancer and his father soon remarried (too hastily in Edmund's opinion) and became more religious and evangelical. This was partly the cause of their quarrels.

Edmund regretted that he did not have the dexterity of his father and he therefore took the greatest pains to write clearly and neatly at all times. Like his father he was meticulous in all that he did and he had strict routines for carrying out everyday tasks. As a young man he started to learn to play the bassoon, and persevered for a year, but reluctantly decided that his coordination was not sufficiently good.

An old schoolfellow recalls that Edmund was one of the most brilliant boys in their school-and also one of the most outrageous. It would seem that he misbehaved partly in rebellion against his father's possessiveness and partly out of boredom. On one occasion he took to school a small electric grill and, in a lesson, started to cook sausages under his desk, soon producing smells of food and loud sizzling noises; his punishment was to work out the cost of the wasted electricity-which was no hardship.

Another characteristic which father and son had in common was a compulsion to hoard. Edmund always worked at a clear desk but he had a 'glory-hole' of a study piled high with bookcases, filing cabinets, drawers and boxes filled with books and papers and potentially useful objects. Among the papers I found just one personal diary kept for just eight months, January to August 1967, nine years before he and I met. Most of the entries contain comments about his work, and included in the period is January 13, the day of what is now the Gibbs derivative, so I have brought with me a copy of that entry.

This brief diary gives a typical slice of Edmund's personal life and a good impression of working life at the National Physical Laboratory at the most fruitful time of his long career there. It was a very busy and hectic time at NPL, with much interchange between the scientists who were all contriving to



Friday JANUARY 13

In the past two weeks, real progress has been made with formulating and codifying the elementary applicable properties of Walsh-Fourier transforms. The watershed came to-day with the discovery of what seemed so difficult yesterday – an operational definition of the logical derivative, rather a mysterious one, but it works. It was reached *via* the Walsh transform of x, which, together with the analogy of the Fourier kernel, suggests that $(k \amalg x)_x = k(k \amalg x)$. This gave a formula for $\{\delta_n(k,0)\}_k$, which was extended to an arbitrary function of k by means of the logical convolution theorem. This formula was then taken as the definition, namely $S'(k) = \frac{1}{2}\sum_{r=0}^{n-1} 2^r\{S(k) - S(k \oplus 2^r)\}$, and the other properties derived. Eight definitions and twelve theorems in four colours make the blackboard very stimulating in appearance. I did some typing for Gebbie. Peter Fraγne was in.

Saturday JANUARY 14

Figure 0.2. A page from the diary of J. Edmund Gibbs with the first formulation of the definition of Gibbs derivatives (Courtesy of Ms. Merion Gibbs).

obtain better computers and to get programs to run. There was much working late and working at weekends, communicating with each other at home, and thinking of ideas in bed at night. Evidently, Edmund got on well with all his colleagues: juniors, his slave-driver of a boss, and the young ladies who regularly brought him flowers for his office desk. He was always interested to exchange a few words with shopkeeper, tradesman or fellow-walker in the park, and he recorded such snippets of conversation.

Edmund was a romantic, and he was interested in classical music, opera, theater and poetry. He wrote poems including a number of love poems; he was very concerned with using different meters and rhyming schemes and he enjoyed writing parody. He recalled with pleasure singing in school and college choirs. In retirement Edmund amused himself making new English translations of the poems of the German Lieder, and Heinrich Hoffmann's Der Struwwelpeter.

One of his girlfriends was from Finland and he studied Finnish and drove to Finland in his small sports car, in winter, to visit her. She married a Finnish diplomat but Edmund kept in touch with her father and, years later, when we married, she met us briefly in London and gave us an ornament of Finnish glass as a present.

Edmund had a remarkable affinity with words; all his thoughts were verbalised. Often, at mealtimes, we solved cryptic crossword puzzles. Between us, at table, we had a small rotating bookcase filled with reference books for checking spellings, pronunciations, alternative meanings and derivations of words, and for looking up literary allusions. When we were tackling the trickiest of the puzzles, he could bring to mind the most unusual and complicated words. I used to say: "Oh, you know everything!", and he would say: "1 knew I'd seen a word for that, I never forget a word-it's just a matter of retrieval". But he could not do anagrams; it was as though the spelling of a word was sacrosanct and there should be no jumbling.

His memory was exceptional in many ways, including a strange facility for forgetting bad things. If I referred to any particularly bad spell of his illness he would say: "Oh, surely not; I have no recollection of that", and he would laugh at me for remembering trivial things and embarrassing moments. He could retain a large amount of information in short term memory. After I retired, we had a routine of walking daily into town to our favourite coffee shop and, if it was quiet there, we read our 'popular science' paperbacks for about an hour. Frequently, on the half-hour walk home, he would relate to me what he had been reading, in detail and in correct order. Odd items of information he did not retain at all; he conducted his daily business by reference to notes. Upstairs and downstairs we each had in- and out-trays and we used to communicate by written notes placed in those trays. He could not remember our car number;

he could not remember our telephone number but he did have a formula for working that out.

One of his most notable characteristics was his single-mindedness. He would not have radio or television on unless he was giving it his whole attention, and he found any interruption of his train of thought very frustrating. If he was expounding a topic and I tried to chip in a comment he would hold up his little finger and plead: "Semi-colon". In company, with several people talking at once, he was quite lost.

Edmund never panicked. If I got excited about something going wrong, he would say: "Enjoy the emergency!". One entry in that diary reads: "Yesterday I received a card from John Chamberlain about a possible error in our paper about which he was anxious. He was to return on Monday or Tuesday, so his concern could have waited." As far as possible, Edmund ignored his illness and did his best to carry on as normal. He studied his ailments and selected the most positive line of thought, said there was no use speculating and demanded "No negative thoughts, please".

Edmund was sincere, loyal, clever, witty and fun, and I am privileged to have been close to him.

Marion Gibbs

3. Remembering J. Edmund Gibbs

It was at the Symposium on the Applications of Walsh Functions held in Spring 1970 in Washington D.C. when I met Dr. Edmund Gibbs for the first time. Prof. Dr. Henning Harmuth, at that time a Visiting Research Professor at the EE-Department of the University Maryland, College Park, USA, had me already informed about the work of Dr. Gibbs on Dyadic Analysis, a mathematical field in which I was interested. For the spring term 1970 I got by Prof. Harmuth the invitation to come to the University of Maryland and to fill there the position as a Visiting Assistant Professor and to help him in his research on the applications of Walsh functions.

Figure 0.3. Dr. Henning F. Harmuth (about 1970)

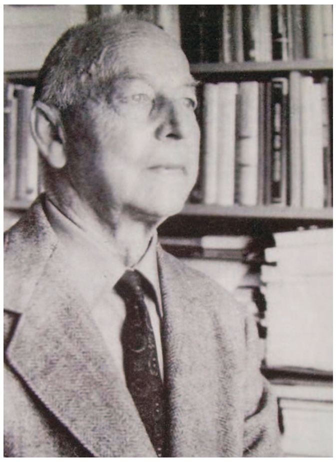


Figure 0.4. Professor Joseph L. Walsh (1970)

At that time, I felt to be a mathematician, however by the need of the EE-Department, headed by Prof. Nicolas DeClaris, I had to run in the spring term 1970 a laboratory course on Electrical Power Engineering and also a laboratory on electrical circuits. By the help of my teaching assistants, who already had experience from earlier such courses, I "survived" and I got even a good evaluation by the students, a kind of success for a mathematician doing practical work in Electrical Engineering. From the teaching assistants, I remember Mr Chang best. He, as I learned, made later a career at the University of Seoul, Korea.

But also my colleagues, especially Prof. Simon, gave me advises and help. In research I continued my work on the theory of dyadic invariant linear systems, Walsh Fourier spectral representations of signals and systems and I did also study of the "logical differentiation" which was introduced by Dr. Gibbs. The results which I achieved at that time got documented in two *Technical Research Reports of the EE-Department*.

A highlight of my stay, besides of a visit of Dallas/Texas for a workshop and on the way back a stop in the fabulous New Orleans/Louisiana, was naturally the Washington Symposium on Walsh Functions in April 1970. It was held at the Naval Research Laboratory and there, as I remember, by the given rules the control of the participants from non-Nato countries were very strict.

As a social highlight the European participants were invited for a dinner to the home of Prof. Joseph Walsh, then already in the age of 75. Since 1965, after his long career at Harvard from 1915-1965, Prof. Walsh lived in the Washington area and was still teaching at the University of Maryland. Prof Walsh was very happy that his paper from of the year 1923, in which he introduced his functions to Mathematics, got now so much attention by communication engineers. He showed us also at his house a practical application of Walsh functions for daily needs: He was wearing a pair of socks ornamented in colour by Walsh functions.

The "Walsh dinner" was also attended by Dr. Edmund Gibbs. By his kind of humour I am sure that Dr. Gibbs liked the socks of Prof. Walsh. This was the first time that Dr. Gibbs and I had some discussions and learned so to know each other better.

After the extension of my visit to Maryland to continue the research on dyadic correlation analysis until the end of August 1970, I returned to my family in Linz and to the University of Linz, where I had a permanent position as a Assistant Professor at the Institute of Mathematics.

It was by a proposal of Dr. Gibbs that in fall 1970 the National Physical Laboratory (NPL), located in Teddington, Middlesex, in close neighbourhood of London, invited me to present a lecture there, and I was very happy to accept this invitation. My wife Ilse accompanied me and we drove with our VW-bug from Linz to London. Dr. Gibbs and the NPL, Division of Electrical Science headed by Mr. Bail, received us with full hospitality. The following remembering on this visit will be not forgotten by me:

1 My lecture was scheduled for 2 p.m. a very bad time for listeners to concentrate. By this time, or by the content of my lecture one elderly gentleman got asleep and felt from his chair. Later he made everything good by his active participation in the discussion.

2 When Dr. Gibbs saw our VW car he admired it fully and made us feel proudly. Later we discovered that he was in the possession of a silver-grey Jaguar sports-car which was parking outside of NPL in order to make his colleagues there not jealous.

Figure 0.5. Edmund Gibbs (Crown Copyright 1970)

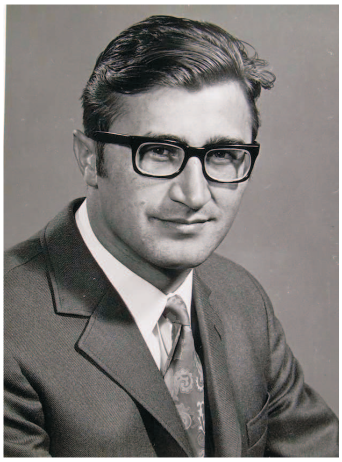


Figure 0.6. Franz Pichler (Crown Copyright, 1970)

3 When we arrived Dr. Gibbs called immediately for the NPL photographer to make pictures of me but also of himself. But not enough, he wanted also that we posted together having punched computer-tapes around our neck. Also pictures where we had to hold models of discrete structures in our hand had to be taken.

4 For the evening Dr. Edmund Gibbs invited us for an event but did not tell us what it would be. It turned out that he had tickets for us to see and listen to the opera Carmen. My wife was extremely fond of that, since so famous artists such as Joan Sutherland and Mario Del Monaco were acting.

In spring 1971, I returned to NPL to accept a Visiting Research Fellowship to stay there for one month. The goal was there to work together with Dr. Gibbs on a common book on the subject of Walsh functions and Dyadic Analysis. However to my surprise, upon arrival, Dr. Gibbs told me that he was ordered that firstly we had to work on a report of the application of Systems Theory in electrical measurements. This work took unfortunately most of the time of my stay and there was not enough time left for the preparation of the planned book. The result was just a Memorandum in which the content of the planned monograph with title "Theory of Bit Processes: A mathematical introduction" were covered.

I returned to Linz joined by Dr. Gibbs since he was supposed to give a lecture at Linz University. We drove with his Jaguar sports-car all the way down from London to Linz. This took us two full days, mainly because Dr.

Figure 0.7. Edmund Gibbs and Franz Pichler struggling with computer tapes (Crown Copyright 1970)

Gibbs had to make a stop every hour to have a cup of tea. For that we usually had to leave the highway to find a restaurant in a nearby town. At the University of Linz the lecture of Dr. Gibbs was received with interest. At that time I was there a member of the Institute of Mathematics and Walsh functions. Besides of my interest in Walsh functions also my colleague Prof. Peter Weiss did some research in that field. We both were with the group of Walsh functions research founded in the 60t's at the University of Innsbruck by Roman Liedl. In Linz, I was at that time associated to the late Prof. Dr. Hans Knapp, who came also from Innsbruck. Knapp was in Innsbruck a member of the research group of Prof. Wolfgang Groebner doing important work in numerical analysis on the basis of Lie-series. Knapp and I showed Dr. Gibbs around the old city of Linz and we visited the different historical buildings and monuments which have a relation to Johannes Kepler, the most famous astronomer which lived in Linz during the time from 1612 to 1628. At the planetary fountain at the Linzer Landhaus, the old government building of the county of Upper Austria, Dr. Gibbs took pictures with his camera. He asked Knapp to move forward and again backwards until he found the right position for taking the picture. Later, when we received the picture from him it turned out that Prof. Knapp was put by him to a position such that it looked that a stream of water from the fountain poured out from his nose. As far as I remember, Prof. Knapp did not like this kind of humour of Edmund Gibbs.

Another point of our sightseeing program was the visit of the Grottenbahn and the Fairy Tale Land which is established in a artificial cave of the old fortress up at the Poestlingberg, the landmark of Linz. We drove together with the children by the little train which is pulled by a dragon around the cave and inspected the different German fairy tales as " Snow White" , and others such as "Cinderella","Hansel and Gretel","Rumplestiltskin" all displayed by fantastic panoramic views.

As far as I remember, Edmund, by his knowledge of the German language and German culture, knew all about it and enjoyed our tour very much. On our way to the Poestlingberg we had to cross the railway track of the local Linz-Aigen-Schlägl line, which connects Linz with the beautiful highland of the Oberes Mühlviertel in the northern part of Austria where also the famous "Böhmerwald" (bohemian woods) is situated reaching far into the part of South-Bohemia, today a part of the Czech Republik. We had to stop at the crossing since a train was coming. There a wagon showed the sign ROTTENEGG, which is the name of one of the next villages on the way. When Edmund saw this, he would say "Oh, sensational, a wagon full of rotten eggs". Certainly I had to drive Edmund next day to Rottenegg and he took many pictures there. It was wise, I guess, that I did not mention another village nearby with the funny name "Huehnergeschrei" which means in English language "chicken-cry", since, I am sure, he would have wanted to go there, to listen there to crying chickens.

Since the time of my close friendship with Dr. Edmund Gibbs, many years have passed. From 1973 on I took the appointment at Linz University as a Professor for Systems Theory at the Department of Computer Science and I left the Institute of Mathematics. Naturally I had to put my interest in Walsh functions aside and to concentrate to the field of Systems Theory, a specific field of Applied Mathematics. So I lost also the close contact to Dr. Edmund Gibbs, a fact which I regret today that it happened. I owe Edmund Gibbs a great deal for his cooperation and friendship. It helped me in that time to get international experience and to start my career as a Professor. My stay at the famous NPL, where Alan Turing was holding once a position, but also in recent time Donald Davies, well known as the inventor of packet switching and for his research in the field of Cryptography, had his office, kept always a place in my CV and I am thankful to have a remembering on the good time there together with Dr. Edmund Gibbs.

In this year 40 years of the existence of the field of Walsh and Dyadic Analysis is celebrated at the Department of Computer Science, Faculty of Electronics of the University of Niš in Serbia by an international workshop. Unfortunately Edmund Gibbs is not with us anymore. We all regret this and I feel sad about

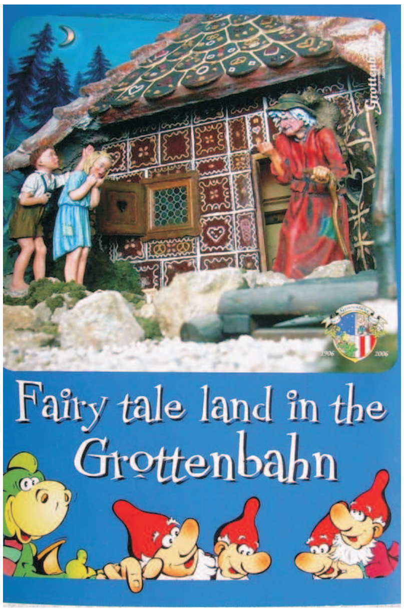


Figure 0.8. Fairy Tale Land in the Grottenbahn at the Pöstlingberg in Linz, Austria

that. His work, however, is with us and will give also in the future many opportunities for interesting mathematical research and related applications in Science and Engineering.

Franz Pichler,
Professor Emeritus
Johannes Kepler University Linz
A-4040 Linz, Austria

Chapter 1

FOURIER ANALYSIS IN A SPACE OF CHARACTERISTIC FUNCTIONS OF SUBSETS OF THE RATIONAL NUMBERS

James Edmund Gibbs

This chapter presents a paper by Dr. J. Edmund Gibbs written in 2004. The paper is reproduced as originally written and typeset by Dr. Gibbs himself.

FOURIER ANALYSIS IN A SPACE OF CHARACTERISTIC FUNCTIONS OF SUBSETS OF THE RATIONAL NUMBERS

J. E. GIBBS

ABSTRACT. The conventional dyadic algebra $\mathbb{F}$ (as in [3]) may be notated as a division algebra over the field of two elements, with the integer singletons as basal units. The space of characteristic functions of the elements of $\mathbb{F}$ admits an analogue of Fourier analysis, extended here to the function space induced similarly by the $\mathbb{F}$-like algebra having the rational singletons as basal units.

1. INTRODUCTION: THE REAL DYADIC ALGEBRA

The function space of the title is isomorphic to the vector space underlying a division algebra $\mathbb{U}$ whose elements are linear combinations

$$A = \sum_{x\in\mathbb{R}} \varphi_A(x)\,\{x\}$$

of real singletons $\{x\}$ with coefficients

$$\varphi_A(x) \in \Phi := \{\varnothing, \{0\}\}\,.$$

The *sum* $A+B$ of $A, B \in \mathbb{U}$ is defined as the symmetric difference

$$A + B = A \vartriangle B = (A \cup B) - (A \cap B)\,.$$

Such summation is extensible to a finite set of addends, and even to convergent infinite series of terms in $\mathbb{U}$. In particular, the sum of a set of pairwise disjoint elements of $\mathbb{U}$ is equal to their union. Thus, for example,

$$\sum_{x\in A} \{x\} = \bigcup_{x\in A} \{x\} = A.$$

The *product* of basal units $\{x\}, \{y\}$ is defined by

$$\{x\}\,\{y\} = \{x+y\}\,.$$

We also define

$$\{x\}\,\varnothing = \varnothing\varnothing = \varnothing\,\{x\} = \varnothing.$$

With the addition and multiplication thus defined, Φ is isomorphic to the field of two elements. Since

$$\sum_{x\in A}\{x\} = A = \sum_{x\in\mathbb{R}} \varphi_A(x)\,\{x\} = \sum_{\varphi_A(x)=\{0\}} \varphi_A(x)\,\{x\} = \sum_{\varphi_A(x)=\{0\}} \{x\}$$

and the set $\{\{x\} : x \in \mathbb{R}\}$ is linearly independent, we have

$$x \in A \quad \text{iff} \quad \varphi_A(x) = \{0\}\,.$$

Thus $\varphi_A(x)$ is the value at x of the *characteristic function* $\varphi_A : \mathbb{R} \to \Phi$ of A.

The function $\varphi : 2^{\mathbb{R}} \to \Phi^{\mathbb{R}}$ that takes each $A \subseteq \mathbb{R}$ to its characteristic function φ_A is a bijection. Its inverse $\varphi^{-1} : \Phi^{\mathbb{R}} \to 2^{\mathbb{R}}$ takes φ_A to

$$A = \{x \in \mathbb{R} : \varphi_A(x) = \{0\}\}.$$

The characteristic function φ_{A+B} satisfies

$$\sum_{x\in\mathbb{R}} \varphi_{A+B}(x)\{x\} = A + B = \sum_{x\in\mathbb{R}} \varphi_A(x)\{x\} + \sum_{x\in\mathbb{R}} \varphi_B(x)\{x\},$$

which implies, by linear independence, that

$$\varphi_{A+B}(x) = \varphi_A(x) + \varphi_B(x).$$

Thus the sum of the subsets A, B of $\mathbb{R}$ is mapped by φ to the pointwise sum $\varphi_A + \varphi_B$ of their characteristic functions.

The function φ_{AB} satisfies

$$\sum_{x\in\mathbb{R}} \varphi_{AB}(x)\{x\} = AB = \sum_{x\in\mathbb{R}} \varphi_A(x)\{x\} \sum_{y\in\mathbb{R}} \varphi_B(y)\{y\}.$$

Since $\sum_{x\in\mathbb{R}} \varphi_A(x)\{x\} = \sum_{x\in\mathbb{R}} \varphi_A(x-y)\{x-y\}$, then,

$$\sum_{x\in\mathbb{R}} \varphi_{AB}(x)\{x\} = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y)\{x-y\}\{y\} = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y)\{x\}.$$

It follows that, at least formally,

$$\varphi_{AB}(x) = \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y).$$

Thus the product AB is mapped by φ to the convolution product $\varphi_A * \varphi_B$, where the convolution operator $\cdot * \cdot : (\Phi^{\mathbb{R}})^2 \to \Phi^{\mathbb{R}}$ is defined formally by

$$(\chi * \psi)(x) = \sum_{y\in\mathbb{R}} \chi(x-y)\psi(y).$$

If, for some $x \in \mathbb{R}$, the sum expressing $\varphi_{AB}(x)$ has infinitely many terms equal to $\{0\}$, then $\varphi_{AB}(x)$ is not determinate. In this case, φ_{AB} is determined, at best, on a proper subset of $\mathbb{R}$; therefore AB is not determinate. If A, B are to belong to an algebra $\mathbb{U}$, then, in particular, their product must be defined. This condition will be satisfied if we stipulate that, for each $A, B \in \mathbb{U}$ and each $x \in \mathbb{R}$, the sum expressing $\varphi_{AB}(x)$ shall have only finitely many non-empty terms. Since

$$\sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y) = \sum_{y\in B\cap[\inf B,\, x-\inf A]} \varphi_A(x-y),$$

there are at most as many such terms as there are elements in the set

$$B \cap [\inf B, x - \inf A],$$

which is finite if each non-empty $A \in \mathbb{U}$ satisfies these conditions:

(i) A is bounded below;
(ii) A has finite intersection with each finite interval.

An immediate consequence of these conditions is that each non-empty A contains its infimum. It turns out that (i) and (ii) are even sufficient to ensure that $\mathbb{U}$ is a division algebra. Thus we are led to

Definition 1.1. *The* ***real dyadic algebra*** $\mathbb{U}$ *is the set*

$$\{\varnothing\} \cup \{A \subset \mathbb{R} : \text{ for some } a_0 \in A,\ a_0 = \inf A \text{ and } \text{for each } \xi \in \mathbb{R},\ A \cap [a_0, \xi] \text{ is finite}\}$$

with the addition and multiplication defined by

$$\varphi_{A+B} = \varphi_A + \varphi_B, \quad \varphi_{AB} = \varphi_A * \varphi_B.$$

The binary operations in $\mathbb{U}$ are defined here in terms of characteristic functions rather than in terms of the corresponding elements of $\mathbb{U}$, as typified by A in the definition of the underlying set. It is not necessary, however, to appeal to the characteristic functions for such definitions: $A+B$ has been defined as $A \vartriangle B$; and

$$AB = \sum_{a\in\mathbb{R}} \varphi_A(a)\,\{a\} \sum_{b\in\mathbb{R}} \varphi_B(b)\,\{b\} = \sum_{a\in A}\sum_{b\in B} \{a+b\}\,.$$

The discussion preceding Definition 1.1 shows that, for each $A, B \in \mathbb{U}$, the product AB is defined (as also, of course, is $A+B$). In verifying that $\mathbb{U}$ is indeed an algebra, we shall next check that $A+B, AB \in \mathbb{U}$. If $A = B$, then $A+B = \varnothing \in \mathbb{U}$. We have $A + \varnothing = A \in \mathbb{U}$, $\varnothing + B = B \in \mathbb{U}$. If $\varnothing \neq A \neq B \neq \varnothing$, then

$$\inf(A+B) \geqq \inf\{\inf A, \inf B\}$$

and

$$\begin{aligned}(A+B) \cap [\inf(A+B), \xi] &\subseteqq (A \cup B) \cap [\inf(A+B), \xi] \\ &\subseteqq (A \cap [\inf A, \xi]) \cup (B \cap [\inf B, \xi])\,,\end{aligned}$$

which is finite; so $A+B \in \mathbb{U}$. If $A = \varnothing$ or $B = \varnothing$, then $AB = \varnothing \in \mathbb{U}$. If $A, B \neq \varnothing$, then

$$\inf AB = \inf A + \inf B,$$

and, with $a_0 := \inf A$, $b_0 := \inf B$,

$$AB \cap [\inf AB, \xi] \subseteqq [a_0 + b_0, \xi] \cap \sum_{a\in A\cap[a_0,\xi-b_0]} \ \sum_{b\in B\cap[b_0,\xi-a_0]} \{a+b\}\,,$$

which is finite; so $AB \in \mathbb{U}$. Thus addition and multiplication are closed in $\mathbb{U}$. This ensures that the following verifications of the remaining field axioms are not merely formal; in particular, summations formally over $\mathbb{R}$ are actually finite.

Addition and multiplication are commutative:

$$\varphi_{A+B} = \varphi_A + \varphi_B = \varphi_B + \varphi_A = \varphi_{B+A},$$

$$\varphi_{AB}(x) = \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y) = \sum_{y\in\mathbb{R}} \varphi_B(x-y)\varphi_A(y) = \varphi_{BA}(x);$$

and associative:

$$\varphi_{A+(B+C)} = \varphi_A + \varphi_{B+C} = \varphi_A + \varphi_B + \varphi_C = \varphi_{A+B} + \varphi_C = \varphi_{(A+B)+C},$$

$$\begin{aligned}\varphi_{A(BC)}(x) &= \sum_{y\in\mathbb{R}}\sum_{z\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y-z)\varphi_C(z) \\ &= \sum_{y\in\mathbb{R}}\sum_{z\in\mathbb{R}} \varphi_A(x-z-y)\varphi_B(y)\varphi_C(z) \\ &= \varphi_{(AB)C}(x);\end{aligned}$$

and multiplication is distributive over addition:

$$\begin{aligned}\varphi_{A(B+C)}(x) &= \sum_{y\in\mathbb{R}} \varphi_A(x-y)\left(\varphi_B(y)+\varphi_C(y)\right)\\ &= \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_B(y) + \sum_{y\in\mathbb{R}} \varphi_A(x-y)\varphi_C(y)\\ &= \varphi_{AB+AC}(x).\end{aligned}$$

The zero of $\mathbb{U}$ is $\varnothing$, and the unity is $\{0\}$. Each $A \in \mathbb{U}$ is its own additive inverse; we shall not have occasion to use the expression $-A$. The existence of the multiplicative inverse $A^{-1} \in \mathbb{U}^*$ of each $A \in \mathbb{U}^* := \mathbb{U} - \{\varnothing\}$ may be proved thus:

If $A \in \mathbb{S}^* := \{S \in \mathbb{U}^* : \text{for some } x \in \mathbb{R},\ S = \{x\}\}$, then $A = \{\inf A\}$ and hence $A^{-1} = \{-\inf A\} \in \mathbb{S}^* \subset \mathbb{U}^*$. If $A \in \mathbb{U}^* - \mathbb{S}^*$, then we may write

$$A = \{a_0, a_1, \ldots\} \quad (a_0 < a_1 < \ldots)$$

and $A = \{a_0\}\, T$, where

$$T = \{a_0\}^{-1} A = \{-a_0\} A \in \mathbb{T} := \{T \in \mathbb{U}^* : \inf T = 0\}.$$

The notation assumes that A (and therefore T) is infinite, but it is not intended to exclude the case of finite A, T.

T^{-1} may be expressed as a (convergent) infinite product, using the notation of a *mixed product* $(0,\infty) \times \mathbb{U} \to \mathbb{U}$ defined by

$$xA = \{\xi : \text{ for some } \eta \in A,\ \xi = x\eta\}.$$

We shall show that

$$T^{-1} = \prod_{r=0}^{\infty} (2^r T) = \{0, t_1, \ldots\}\{0, 2t_1, \ldots\}\ldots \in \mathbb{T} \subset \mathbb{U}^*.$$

Convergence [1] of such an infinite product may be proved by using

Lemma 1.1. *A sequence (P_n) of subsets of $\mathbb{R}$ converges iff for some (C_n),*

$$C_n \subseteqq C_{n+1}\ (n \in \mathbb{P}), \quad \lim_{n\to\infty} C_n = \mathbb{R}, \quad C_n \cap P_\nu = C_n \cap P_n\ (\nu \geqq n).$$

If such a (C_n) exists, then $(C_n \cap P_n)$ converges, and

$$P := \lim P_n = \lim C_n \cap P_n.$$

Proof. If there is a (C_n) satisfying the proposed conditions, then

$$\bigcup_{n=1}^{\infty} C_n = \lim C_n = \mathbb{R}.$$

So, for each $x \in \mathbb{R}$, for some $m \in \mathbb{P}$, we have $x \in C_m$. Let $n(x)$ denote the least such m. Then $x \in C_{n(x)}$ and for each $\nu \geqq n(x)$,

$$\varphi_{P_\nu}(x) = \varphi_{C_{n(x)} \cap P_\nu}(x) = \varphi_{C_{n(x)} \cap P_{n(x)}}(x).$$

Thus the sequence $\left(\varphi_{P_\nu}\right)$ of characteristic functions converges to φ_P, defined by

$$\varphi_P(x) = \lim_{\nu\to\infty} \varphi_{P_\nu}(x) = \varphi_{C_{n(x)} \cap P_{n(x)}}(x).$$

Consequently (P_n) converges. $(C_n \cap P_n)$ also converges, for

$$C_n \cap P_n = C_n \cap P_{n+1} \subseteqq C_{n+1} \cap P_{n+1}.$$

It follows that

$$P = \mathbb{R} \cap P = \lim C_n \cap \lim P_n = \lim (C_n \cap P_n)\,.$$

Again, if (P_n) converges to P, then (φ_{P_n}) converges pointwise to φ_P, defined by

$$\varphi_P(x) = \lim_{\nu \to \infty} \varphi_{P_\nu}(x).$$

For each $n \in \mathbb{P}$, let

$$C_n := \{x : \text{ for each } \nu \geqq n,\ \varphi_{P_\nu}(x) = \varphi_{P_n}(x)\}\,.$$

Then $C_n \cap P_\nu = C_n \cap P_n$ $(\nu \geqq n)$; and $C_n \subseteqq C_{n+1}$, so that (C_n) converges to

$$C := \bigcup_{n=1}^{\infty} C_n \subseteqq \mathbb{R}.$$

Since (φ_{P_ν}) converges, for each $x \in \mathbb{R}$ there is an $n \in P$ such that $x \in C_n$. Hence $\mathbb{R} \subseteqq C$; so $\lim C_n = C = \mathbb{R}$. Thus (C_n) meets the proposed conditions. □

To prove that the infinite product expression of T^{-1} is convergent, we apply Lemma 1.1 to the sequence (P_n) of partial products, where

$$P_n := \prod_{r=0}^{n-1} (2^r T) = \{0, t_1, \ldots\} \ldots \{0, 2^{n-1}t_1, \ldots\}\,.$$

If the sequence (C_n) is defined by $C_n = (-\infty, 2^n t_1)$, then (C_n) satisfies the conditions of the lemma. Thus (P_n) converges and

$$P := \prod_{r=0}^{\infty} (2^r T) = \lim_{n \to \infty} P_n = \lim_{n \to \infty} \Big((-\infty, 2^n t_1) \cap \prod_{r=0}^{n-1} (2^r T)\Big).$$

To show that $P \in \mathbb{T}$, we note that $\inf P = 0$ and that

$$P \cap [0, \xi] \subseteqq P \cap C_{n(\xi)} = P_{n(\xi)} \cap C_{n(\xi)} = P_{n(\xi)} \cap [0, 2^{n(\xi)} t_1),$$

which is finite since $P_{n(\xi)} \in \mathbb{U}$. That $P = T^{-1}$ is seen by using the formula

$$T^{2^r} = 2^r T \quad (T \in \mathbb{T} := \{T \in \mathbb{U} : \inf T = 0\}\,,\ r \in \mathbb{Z})\,,$$

which may be proved by observing that

$$T^2 = \sum_{x \in T,\, y \in T,\, x < y} \{x + y\} + \sum_{x \in T,\, y \in T,\, x > y} \{x + y\} + \sum_{x = y \in T} \{x + y\}\,.$$

Renotation of (x, y) as (y, x) shows that

$$\sum_{x \in T,\, y \in T,\, x < y} \{x + y\} = \sum_{y \in T,\, x \in T,\, y < x} \{y + x\} = \sum_{x \in T,\, y \in T,\, x > y} \{x + y\}$$

and therefore

$$T^2 = \sum_{x = y \in T} \{x + y\} = \sum_{x \in T} \{2x\} = 2T.$$

It follows by induction that $T^{2^r} = 2^r T$ $(r \in \mathbb{N})$. This result is extended to negative r by using the left-associativity of the mixed product (an immediate consequence of the definition): thus, for each $r \in \mathbb{N}$,

$$T^{2^{-r}} = 2^{-r} 2^r T^{2^{-r}} = 2^{-r} T^{2^{-r} 2^r} = 2^{-r} T.$$

So $T^{2^r} = 2^r T \ (r \in \mathbb{Z})$. Two applications of this equality yield

$$TP_n = T \prod_{r=0}^{n-1} (2^r T) = T \prod_{r=0}^{n-1} T^{2^r} = T^{2^n} = 2^n T = \{0, 2^n t_1, \ldots\}.$$

Hence, as in the proof of Lemma 1.1,

$$\varphi_{TP}(x) = \varphi_{C_{n(x)} \cap TP_{n(x)}}(x) = \varphi_{(-\infty, 2^{n(x)} t_1) \cap 2^{n(x)} T}(x) = \varphi_{\{0\}}(x),$$

which implies that $TP = \{0\}$, so that $T^{-1} = P$. It follows that

$$A^{-1} = \{a_0\}^{-1} T^{-1} = \{-a_0\} P = \{-a_0\} \prod_{r=0}^{\infty} (2^r (\{-a_0\} A)) \in \mathbb{U}.$$

We have proved that $\mathbb{U}$ is a field under the addition and multiplication proposed in Definition 1.1. Since Φ is a subfield of $\mathbb{U}$, the additive group $\mathbb{U}$, equipped with the restriction, to $\Phi \times \mathbb{U}$, of the multiplication in $\mathbb{U}$, is a vector space over Φ. For the same reason, this vector space, equipped with the multiplication in $\mathbb{U}$, is a commutative division algebra over Φ.

The conventional dyadic algebra $\mathbb{F}$ [3] is clearly isomorphic to the subalgebra

$$\mathbb{U}_{\mathbb{Z}} := \{A \in \mathbb{U} : A \subset \mathbb{Z}\}$$

of $\mathbb{U}$.

2. Rational Dyadic Exponentiation

To facilitate our treatment, in Section 3, of harmonic analysis in the subspace $\mathbb{U}_{\mathbb{Q}}$ of $\mathbb{U}$, we introduce, beyond addition and multiplication, a third operation:

Definition 2.1. ***Rational dyadic exponentiation*** *is the operation of raising a* ***base*** $B \in \mathbb{U}$ *to a* ***power*** $B \uparrow X \in \mathbb{U}$, *where the* ***exponent***

$$X \in \mathbb{U}_{\mathbb{Q}} := \{A \in \mathbb{U} : A \subset \mathbb{Q}\}.$$

The operator $\cdot \uparrow \cdot : \mathbb{U} \times \mathbb{U}_{\mathbb{Q}} \to \mathbb{U}$ *is defined by*

$$B \uparrow X = \sum_{x \in X} B^x$$

iff both the following conditions are satisfied:

$$\begin{array}{ll} \text{(i)} & B \neq \varnothing \ \textit{ or } \ \inf X \geqq 0; \\ \text{(ii)} & X \textit{ is finite } \ \textit{or} \ \inf B > 0. \end{array}$$

Condition (i) excludes the occurrence of $\varnothing^x$ $(x < 0)$ as a term in the defining sum. Condition (ii) ensures convergence of $\sum_{x \in X} B^x$ to an element of $\mathbb{U}$.

It may be proved that for each

$$(B, x) \in \mathbb{U} \times \mathbb{Q} - \{\varnothing\} \times (-\infty, 0),$$

there is a unique $B^x \in \mathbb{U}$. It is not excluded that B^x takes one or more other values not in $\mathbb{U}$. Such failure of B^x to be single-valued does occur, but it is convenient to confine attention to the unique branch of this function taking values in $\mathbb{U}$. In this paper, then, the notation B^x will be understood to refer to this branch alone. We shall obtain an explicit expression (as a convergent infinite product) for the general rational power $B^{m/n}$ $(B \in \mathbb{U}, m \in \mathbb{Z}, n \in \mathbb{P}, (m, n) = 1)$. From this it will be clear that we have seized the intended branch of the function, that taking values in $\mathbb{U}$.

To reduce $B^{m/n}$ to a more convenient form, we define

$$T = \{-\inf B\}\, B \in \mathbb{T}.$$

Then $B = \{\inf B\}\, T$; so

$$B^{m/n} = \{\inf B\}^{m/n}\, T^{m/n} = \{(m/n) \inf B\}\, T^{m/n}.$$

A more manageable form of the exponent is

$$m/n = 2^k p/q,$$

where $k, p \in \mathbb{Z},\ q \in \mathbb{P}$, each of p, q is odd, and $(p, q) = 1$. These conditions determine k, p, q uniquely.

Using the canonical (monomorphic) embedding of the rational integers $\mathbb{Z}$ into the 2-adic integers $\mathbb{Z}_2$, and the fact [2] that the odd integers are invertible in $\mathbb{Z}_2$, we write $r := p/q$ as a 2-adic integer (in the notation of formal series)

$$\sum_{i \in \mathbb{N}} r_i 2^i = r = p/q = \sum_{i \in \mathbb{N}} p_i 2^i \Big/ \sum_{j \in \mathbb{N}} q_j 2^j.$$

Then, at least heuristically,

$$T^{p/q} = T^r = T^{\sum_{i \in \mathbb{N}} r_i 2^i} = \prod_{i \in \mathbb{N}} T^{r_i 2^i} = \prod_{i \in \mathbb{N}} (2^i T)^{r_i}.$$

Convergence of the infinite product to an element P of $\mathbb{U}$ is proved by writing

$$T = \{0, t_1, \ldots\} \quad (0 < t_1 < \ldots)$$

and applying Lemma 1.1, with $C_n = (-\infty, 2^n t_1)$, to the sequence of partial products

$$P_n := \prod_{i=0}^{n-1} (2^i T)^{r_i} = \{0, t_1, \ldots\}^{r_0} \ldots \{0, 2^{n-1} t_1, \ldots\}^{r_{n-1}},$$

as in the proof given earlier of the convergence of $T^{-1} = \prod_{i=0}^{\infty} (2^i T)$. Convergence of the infinite product P validates the assumption above that

$$T^{\sum_{i \in \mathbb{N}} r_i 2^i} = \prod_{i \in \mathbb{N}} T^{r_i 2^i}.$$

The formal series $r = p/q$ is computed by division of the series p by the series q, as in the following numerical illustration.

We shall calculate

$$B^{m/n} = \{1, 2, 8\}^{-3/10},$$

an easy example, since B is finite and has integer elements. We note first that

$$B^{m/n} = \{-3/10\}\, \{0, 1, 7\}^{2^{-1}(-3/5)} = \{-3/10\}\, (2^{-1}\, \{0, 1, 7\}^{-3/5})$$

and we evaluate $\{0, 1, 7\}^{-3/5}$. As a 2-adic integer,

$$-3 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + \ldots.$$

For brevity, we write this in *conventional* binary notation continued indefinitely *leftwards* from the (suppressed) binary point, with the leftwards recurring period $\ldots 1$ written as $\overline{1}$. In this notation, $-3 = \overline{1}01,\ 5 = \overline{0}101$, and the quotient

$$-3/5 = \overline{1}01/\overline{0}101 = \ldots 10011001 = \overline{1001}$$

is obtained by left-going long division, starting at the right. Hence

$$\{0,1,7\}^{-3/5} = \{0,1,7\}^{\overline{1001}} = \{0,1,7\}^{2^0}\{0,1,7\}^{2^3}\{0,1,7\}^{2^4}\ldots;$$

therefore

$$\begin{aligned}\{0,1,7\}^{-3/5} &= (2^0\{0,1,7\})(2^3\{0,1,7\})(2^4\{0,1,7\})\ldots\\ &= \{0,1,7\}\{0,8,56\}\{0,16,112\}\ldots\\ &= \{0,1,7,8,9,15,56,57,63\}\{0,16,112\}\ldots\\ &= \{0,1,7,8,9,15,16,17,23,24,25,31,\ldots\};\end{aligned}$$

so

$$\{1,2,8\}^{-3/10} = \{-3/10\}\,(2^{-1}\{0,1,7,8,9,15,16,17,23,24,25,31,\ldots\}).$$

The following properties of rational dyadic exponentiation $\cdot \uparrow \cdot$ will prove helpful in our development of harmonic analysis in $\mathbb{U}$:

Lemma 2.1. *For each* $B \in \mathbb{U}^*$ $(\inf B > 0)$, *the function* $\beta_B : \mathbb{U}_\mathbb{Q} \to \mathbb{U}$ *defined by*

$$\beta_B(X) = B \uparrow X$$

is a field monomorphism.

Proof. The conditions $B \in \mathbb{U}^*$, $\inf B > 0$ ensure that expressions entering this proof exist and converge. We omit the details. By an obvious generalisation of the equalities

$$\sum_{a\in X+Y}\{a\} = \sum_{x\in X}\{x\} + \sum_{y\in Y}\{y\}, \qquad \sum_{a\in XY}\{a\} = \sum_{x\in X}\sum_{y\in Y}\{x+y\},$$

we obtain

$$B\uparrow(X+Y) = \sum_{a\in X+Y} B^a = \sum_{x\in X} B^x + \sum_{y\in Y} B^y = B\uparrow X + B\uparrow Y,$$

$$B\uparrow(XY) = \sum_{a\in XY} B^a = \sum_{x\in X}\sum_{y\in Y} B^{x+y} = \sum_{x\in X} B^x \sum_{y\in Y} B^y = (B\uparrow X)\,(B\uparrow Y).$$

Thus β_B is a field homomorphism. To prove that β_B is injective, suppose that, for some $X, Y \in \mathbb{U}_\mathbb{Q}$, we have $X \neq Y$ and $\beta_B(X) = \beta_B(Y)$. Then

$$\beta_B(X+Y) = \beta_B(X) + \beta_B(Y) = \varnothing.$$

But, by our supposition, $X + Y \neq \varnothing$; so

$$\inf(\beta_B(X+Y)) = \inf\left(\sum\nolimits_{a\in X+Y} B^a\right) = (\inf B)\inf(X+Y).$$

Consequently $(\inf B)\inf(X+Y) \in \beta_B(X+Y)$; hence

$$\beta_B(X+Y) \neq \varnothing.$$

The contradiction shows that β_B is injective, and thus a field monomorphism. □

Lemma 2.2. *For each* $B \in \mathbb{U}^*$, $X \in \mathbb{U}_\mathbb{Q}^* := \mathbb{U}_\mathbb{Q} - \{\varnothing\}$, $Y \in \mathbb{U}_\mathbb{Q}$ $(\inf B, \inf X > 0)$,

$$B\uparrow(X\uparrow Y) = (B\uparrow X)\uparrow Y.$$

Proof. We prove first that for each $B \in \mathbb{U}^*$ $(\inf B > 0)$, $X \in \mathbb{U}^*_{\mathbb{Q}}$, $y \in \mathbb{Q}$,

$$B \uparrow (X^y) = (B \uparrow X)^y.$$

This is done by making the substitutions

$$y = p/q \ \ (p \in \mathbb{Z}, q \in \mathbb{P}, (p,q) = 1)\,, \quad X = Z^q$$

and using Lemma 2.1:

$$\beta_B(X^y) = \beta_B(Z^p) = (\beta_B(Z))^p = ((\beta_B(Z))^q)^y = (\beta_B(Z^q))^y = (\beta_B(X))^y\,.$$

Equivalently, $B \uparrow (X^y) = (B \uparrow X)^y$. It follows that

$$(B \uparrow X) \uparrow Y = \sum_{y \in Y} (B \uparrow X)^y = \sum_{y \in Y} B \uparrow (X^y)\,.$$

If Y is finite, then

$$(B \uparrow X) \uparrow Y = \sum_{y \in Y} B \uparrow (X^y) = B \uparrow \sum_{y \in Y} X^y = B \uparrow (X \uparrow Y)$$

and for this case a proof of associativity is complete. Suppose, then, that Y is infinite. Write $Y = \{y_0, y_1, \ldots\}$ $(y_0 < y_1 < \ldots)$ and we get

$$(B \uparrow X) \uparrow Y = \lim_{n \to \infty} \sum_{r=0}^{n-1} B \uparrow (X^{y_r}) = \lim_{n \to \infty} B \uparrow \sum_{r=0}^{n-1} X^{y_r}.$$

If Y is bounded above by some ξ, then $Y = Y \cap [\inf Y, \xi]$, which, by Definition 1.1, is finite, contrary to supposition. Thus Y is not bounded above; so $\lim y_n = \infty$. For some $n_0 \in \mathbb{N}$, then, for each $n > n_0$, we have $y_n > 0$ and may write

$$y_n = p_n/q_n \ \ (p_n, q_n \in \mathbb{P},\ (p_n, q_n) = 1)\,, \quad X = Z_n^{q_n}.$$

Hence

$$\inf X = q_n \inf Z_n, \quad X^{y_n} = Z_n^{q_n y_n} = Z_n^{p_n}.$$

So, since $X \in \mathbb{U}^*_{\mathbb{Q}}$,

$$\inf (X^{y_n}) = \inf (Z_n^{p_n}) = p_n \inf Z_n = y_n \inf X \in \mathbb{Q}, \ \ \inf (X^{y_n}) > 0.$$

Likewise

$$\inf \left(B^{\inf (X^{y_n})}\right) = \inf (X^{y_n}) \inf B = (y_n \inf X) \inf B.$$

Therefore

$$\inf A_n = \inf \sum_{x \in \sum_{r=n}^{\infty} X^{y_r}} B^x = \inf \left(B^{\inf (X^{y_n})}\right) = (\inf B)\, y_n \inf X.$$

Thus

$$\lim (\inf A_n) = \lim (\inf B)\, y_n \inf X = \infty,$$

which implies that $\lim A_n = \varnothing$. Consequently

$$(B \uparrow X) \uparrow Y = \lim_{n \to \infty} B \uparrow \sum_{r=0}^{n-1} X^{y_r} = \lim A_n + B \uparrow (X \uparrow Y) = B \uparrow (X \uparrow Y).$$

□

3. Fourier Analysis in the Rational Dyadic Space

We have seen that the additive group $\mathbb{U}$, equipped with the restriction to $\Phi \times \mathbb{U}$ of the multiplication in $\mathbb{U}$, is a vector space over Φ. We shall discuss harmonic analysis in the subspace $\mathbb{U}_{\mathbb{Q}}$ of $\mathbb{U}$, beginning with

Definition 3.1. *The* ***Fourier transform operator*** $\widehat{\cdot}: \mathbb{U}_{\mathbb{Q}} \to \mathbb{U}_{\mathbb{Q}}$ *is defined by*

$$\widehat{A} = \mathbb{P} \uparrow A = \sum_{x \in A} \mathbb{P}^x .$$

(The defining series exists and is convergent because $\mathbb{P} \neq \varnothing$ and $\inf \mathbb{P} > 0$. From now on we shall generally not mention such details, leaving them to the reader.)

This apparently arbitrary definition will be given some plausibility by a proof that $\widehat{\cdot}$ is a *self-inverse automorphism* of the field $\mathbb{U}_{\mathbb{Q}}$. (Our analysis differs in this respect from conventional Fourier analysis, whose transform operator is a *skew-inverse isomorphism* which preserves sums and takes pointwise products into convolution products and vice versa. As it were by way of compensation, *dyadic conjugation* is skew-inverse, not self-inverse like its conventional analogue, complex conjugation.)

A proof depends upon the following property of dyadic exponentiation $\cdot \uparrow \cdot$:

$$\mathbb{P} \uparrow \mathbb{P} = \{1\} .$$

To see this, we define $\mathbb{J} = \{0, 1\}$, and we sum the (convergent) geometric series

$$\sum_{r=0}^{\infty} \mathbb{P}^r = (\{0\} + \mathbb{P})^{-1} = \mathbb{N}^{-1} = \mathbb{J}$$

in the usual way. (The inverses quoted may be checked by multiplication.) Then

$$\mathbb{P} \uparrow \mathbb{P} = \sum_{x \in \mathbb{P}} \mathbb{P}^x = \sum_{r=0}^{\infty} \mathbb{P}^{r+1} = \mathbb{P} \sum_{r=0}^{\infty} \mathbb{P}^r = \mathbb{P}\mathbb{J} = \{1\} .$$

The element $\{1\}$ of $\mathbb{U}_{\mathbb{Q}}$ is a (left and right) *exponentiative identity*:

$$\{1\} \uparrow A = \sum_{x \in A} \{1\}^x = \sum_{x \in A} \{x\} = A, \qquad A \uparrow \{1\} = \sum_{x \in \{1\}} A^x = A^1 = A.$$

Using Lemma 2.2, we can prove that the Fourier transform $\widehat{\cdot}$ is self-inverse:

$$\widehat{\;}(\widehat{A}) = \mathbb{P} \uparrow (\mathbb{P} \uparrow A) = (\mathbb{P} \uparrow \mathbb{P}) \uparrow A = \{1\} \uparrow A = A.$$

Since, by definition, $\widehat{A} = \mathbb{P} \uparrow A = \beta_{\mathbb{P}}(A)$, Lemma 2.1 shows that $\widehat{\cdot}$ is monomorphic, and thus a self-inverse automorphism of $\mathbb{U}_{\mathbb{Q}}$.

It will be shown that the Fourier operator $\widehat{\cdot}$ may be viewed as an *orthogonal* transform. Such transforms are conventionally defined on a unitary space (a vector space over $\mathbb{C}$ with an Hermitian inner product). The definition of such an inner product uses complex conjugation. We therefore introduce an analogue $(\cdot)^*$ of the complex conjugation operator $\bar{\cdot}$.

The complex conjugate is already defined on the field $\mathbb{C}$ underlying a unitary space, and trivially induces conjugation of vectors in that space. The meagreness of the field Φ underlying the space $\mathbb{U}_{\mathbb{Q}}$ seems to preclude non-trivial definition of an analogous conjugate on Φ that would induce conjugation of vectors in $\mathbb{U}_{\mathbb{Q}}$. The operator $(\cdot)^*$ is therefore defined directly on (a subset of) $\mathbb{U}_{\mathbb{Q}}$:

Definition 3.2. *The* ***dyadic conjugate*** A^* *of* $A \in \mathbb{U}_{\mathbb{Q}}$ *is defined by*

$$\varphi_{A^*}(x) = \sum_{y \in \widehat{A}} \varphi_{\mathbb{P}^x}(y)$$

iff the summation on the right converges for each $x \in \mathbb{Q}$.

The dyadic conjugate A^* is not defined for every $A \in \mathbb{U}_{\mathbb{Q}}$. For example, $\{1\}^*$ is undefined. Indeed, $\widehat{\{1\}} = \mathbb{P} \uparrow \{1\} = \mathbb{P}$. So, formally,

$$\varphi_{\{1\}^*}(1) = \sum_{y \in \mathbb{P}} \varphi_{\mathbb{P}}(y) = \sum_{r=1}^{\infty} \{0\}.$$

But $(\cdot)^*$ is not empty. Indeed, $\widehat{\mathbb{P}} = \mathbb{P} \uparrow \mathbb{P} = \{1\}$. So, for each $x \in \mathbb{Q}$,

$$\varphi_{\mathbb{P}^*}(x) = \sum_{y \in \{1\}} \varphi_{\mathbb{P}^x}(y) = \varphi_{\mathbb{P}^x}(1).$$

Definition 3.3. *The* ***inner product*** $\langle A, B\rangle \in \Phi$ *of* $A, B \in \mathbb{U}_{\mathbb{Q}}$ *is defined by*

$$\langle A, B\rangle = \sum_{x \in \mathbb{Q}} \varphi_A(x)\varphi_{B^*}(x)$$

iff B^* *is defined and the summation on the right converges.*

The definiens here is only formally a summation over $\mathbb{Q}$: it may be written

$$\sum_{x \in \mathbb{Q}} \varphi_A(x)\varphi_{B^*}(x) = \sum_{x \in A} \varphi_{B^*}(x) = \sum_{x \in B^*} \varphi_A(x).$$

The partial function $\langle \cdot, \cdot \rangle : \mathbb{U}^2 \to \Phi$ is symmetric: $\langle A, B\rangle = \langle B, A\rangle$. Indeed, if $\langle A, B\rangle$ is defined, then so is $B^* = \sum_{y \in \widehat{B}} \varphi_{\mathbb{P}^x}(y)$, which implies that this summation is finite. This justifies the last step in the calculation

$$\langle A, B\rangle = \sum_{x \in A} \varphi_{B^*}(x) = \sum_{x \in A} \sum_{y \in \widehat{B}} \varphi_{\mathbb{P}^x}(y) = \sum_{y \in \widehat{B}} \sum_{x \in A} \varphi_{\mathbb{P}^x}(y).$$

Since $\inf \mathbb{P}^x = x \inf \mathbb{P} = x$, we have $\varphi_{\mathbb{P}^x}(y) = \varnothing$ if $x > y$. So

$$\sum_{x \in A} \varphi_{\mathbb{P}^x}(y) = \sum_{x \in A \cap [\inf A, y]} \varphi_{\mathbb{P}^x}(y),$$

which shows that the summation is finite. By definition of addition, then,

$$\sum_{x \in A} \varphi_{\mathbb{P}^x}(y) = \varphi_{\sum_{x \in A} \mathbb{P}^x}(y) = \varphi_{\mathbb{P} \uparrow A}(y)\varphi_{\widehat{A}}(y).$$

Thus we obtain the $\mathbb{U}_{\mathbb{Q}}$-analogue of *Parseval's theorem:* iff $\langle A, B\rangle$ is defined,

$$\langle A, B\rangle = \sum_{y \in \widehat{B}} \varphi_{\widehat{A}}(y) = \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{B}}(y).$$

The symmetry of the last expression shows that $\langle A, B\rangle = \langle B, A\rangle$.

The function $\langle \cdot, \cdot \rangle$ is also bilinear: for each $\lambda, \mu \in \Phi$,

$$\langle A, \lambda B + \mu C\rangle = \lambda \langle A, B\rangle + \mu \langle A, C\rangle$$

iff $\langle A, B\rangle, \langle A, C\rangle$ are defined. Indeed, the condition mentioned implies that

$$\sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{B}}(y), \ \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{C}}(y)$$

are finite summations. So

$$\lambda \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{B}}(y) + \mu \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{C}}(y) = \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\left(\lambda\varphi_{\widehat{B}}(y) + \mu\varphi_{\widehat{C}}(y)\right).$$

It is easy to see that, for each $A \in \mathbb{U}_{\mathbb{Q}}$, $\lambda\varphi_A(y) = \varphi_{\lambda A}(y)$. Hence

$$\lambda\varphi_{\widehat{B}}(y) + \mu\varphi_{\widehat{C}}(y) = \varphi_{\lambda\widehat{B}}(y) + \varphi_{\mu\widehat{C}}(y) = \varphi_{\lambda\widehat{B}+\mu\widehat{C}}(y) = \varphi_{\widehat{\ }(\lambda B+\mu C)}(y),$$

where the last equality follows from the automorphic property of $\widehat{\cdot}$. Therefore

$$\lambda\langle A, B\rangle + \mu\langle A, C\rangle = \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{\ }(\lambda B+\mu C)}(y) = \langle A, \lambda B + \mu C\rangle.$$

Since Φ is not an ordered field, positive definiteness of the associated quadratic form is out of the question, so the properties of symmetry and bilinearity alone are sufficient justification for calling $\langle\cdot,\cdot\rangle$ an inner product.

As analogues of familiar properties of the conventional Fourier transform,

$$\widehat{\ }(\mathbb{P}^{\xi}) = \mathbb{P} \uparrow (\mathbb{P}^{\xi}) = (\mathbb{P} \uparrow \mathbb{P})^{\xi} = \{1\}^{\xi} = \{\xi\}$$

and, by the self-inverseness of $\widehat{\cdot}$,

$$\widehat{\ }\{\xi\} = \widehat{\ }(\widehat{\ }(\mathbb{P}^{\xi})) = \mathbb{P}^{\xi}.$$

To complete our view of $\widehat{\cdot}$ as an orthogonal transform, we prove

Theorem 3.1. *The set* $\left\{\mathbb{P}^{\xi} : \xi \in \mathbb{Q}\right\}$ *is orthonormal and complete in* $\mathbb{U}_{\mathbb{Q}}$.

Proof. The set $\left\{P^{\xi} : \xi \in \mathbb{Q}\right\}$ is orthonormal: for each $\xi, \eta \in \mathbb{Q}$ $(\xi \neq \eta)$,

$$\left\langle \mathbb{P}^{\xi}, P^{\eta}\right\rangle = \sum_{y \in \mathbb{Q}} \varphi_{\{\xi\}}(y)\varphi_{\{\eta\}}(y) = \sum_{y \in \{\xi\}} \varphi_{\{\eta\}}(y) = \varphi_{\{\eta\}}(\xi) = \varnothing,$$

$$\left\|\mathbb{P}^{\xi}\right\| = \left\langle \mathbb{P}^{\xi}, \mathbb{P}^{\xi}\right\rangle = \varphi_{\{\xi\}}(\xi) = \{0\}.$$

The orthonormal set $\left\{\mathbb{P}^{\xi} : \xi \in \mathbb{Q}\right\}$ is complete in $\mathbb{U}_{\mathbb{Q}}$: if, for each $A \in \mathbb{U}_{\mathbb{Q}}$, $\xi \in \mathbb{Q}$, we have $\left\langle A, \mathbb{P}^{\xi}\right\rangle = \varnothing$, then

$$\varnothing = \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\widehat{\ }(\mathbb{P}^{\xi})}(y) = \sum_{y \in \mathbb{Q}} \varphi_{\widehat{A}}(y)\varphi_{\{\xi\}}(y) = \varphi_{\widehat{A}}(\xi).$$

Hence $\widehat{A} = \varnothing$, which implies that

$$A = \widehat{\ }(\widehat{A}) = \mathbb{P} \uparrow \widehat{A} = \mathbb{P} \uparrow \varnothing = \sum_{x \in \varnothing} \mathbb{P}^{x} = \varnothing.$$

□

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Chapter 2

GIBBS DERIVATIVES - THE DEVELOPMENT OVER 40 YEARS IN CHINA

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Abstract Beginning at 1975, we learned about Gibbs derivatives, since then, we have studied the topics in the Gibbs derivatives, some related properties of this new kind of derivatives and their applications.

In this survey, we arrange the presentation as follows.

1. Generalize Gibbs derivatives to p-adic groups, p_j-adic groups, local fields and locally compact Vilenkin groups; apply to approximation theory by using Gibbs derivatives;
2. Define function spaces on local fields described by Gibbs derivatives;
3. Show comparison between $\mathbf{R}^n$ analysis based on classical calculus and local field analysis based on Gibbs calculus;
4. Establish principles for defining "rate of change";
5. View some applications of Gibbs derivatives to fractal analysis.

1. Generalizations of Gibbs Derivatives and Applications

Since the Gibbs derivatives were born in 1967, many mathematicians and physicists, engineers and technicians have paid great attentions to this topic. And so do we.

We learned about the Gibbs derivatives in 1975. Since then, we have started the study of Gibbs derivatives. We have generalized Gibbs derivatives to p-adic groups, p_j-adic groups, local fields and locally compact Vilenkin groups.

Gibbs derivatives on p-adic groups and p_j-adic groups

In 1978, [1], F.X. Ren, W.Y. Su, W.X. Zheng generalized the dyadic Gibbs derivatives to p-adic case for p as a prime.

Let for

$$G = \{x = (x_{-s}, x_{-s+1}, \cdots, x_{-1}, x_0, x_1, \cdots) : x_j \in \{0, 1, \cdots, p-1\}, s \in \mathbf{P}\}$$

be a p-adic group, with addition coordinately $\oplus \bmod p$ and $\mathbf{P} = \{0\} \bigcup \mathbf{N}$. If we endow a topology $\tau = \{B_k\}_{k \in \mathbf{Z}}$ to G, where

$$B_k = \{x = (x_k, x_{k+1}, \cdots) \in G : x_j \in \{0, 1, \cdots, p-1\}, j \geq k, x_k \neq 0\}$$

is a k-neighborhood of $0 \in G$, then G becomes a non-discrete, totally disconnected, locally compact topological group; and the Walsh system

$$\Gamma = G^{\wedge} = \{w_p(x, y) = \exp \frac{2\pi i}{p} x \otimes y : x, y \in G\}, \quad x \otimes y = \sum x_{1-j} y_j$$

is the character group of G.

If $f : G \to \mathbf{C}$ is a complex valued function on G, we call

$$\begin{aligned} f^{<1>}(x) &= \lim_{N \to +\infty} \Delta_N f(x) \qquad (2.1) \\ &= \lim_{N \to +\infty} \sum_{k=-N}^{N} p^k \sum_{j=0}^{p-1} A_j f(x \oplus j p^{-k-1}), \quad x \in G \end{aligned}$$

p-adic pointwise Gibbs derivative of $f(x)$ at $x \in G$, where

$$A_0(p) = \frac{p-1}{2}, A_j = \frac{\omega^j}{1-\omega^j}, \quad j = 1, \cdots, p-1, \quad \omega = \exp \frac{2\pi i}{p}.$$

One also can define p-adic strong Gibbs derivative of $f(x)$ in $L^r(G)$, $1 \leq r \leq +\infty$.

The special case of (2.1) is for the compact case, $G_0 \subset G$ is the compact subgroup of G

$$G_0 = \{x = (x_0, x_1, \cdots) \in G : x_j \in \{0, 1, \cdots, p-1\}, j \in \mathbf{P}\},$$

and

$$\Gamma_0 = G_0^{\wedge} = \{w_p(k, x) = \exp \frac{2\pi i}{p} k \otimes x : x \in G, k \in \mathbf{P}\}.$$

Then the Gibbs derivatives of $f(x)$ at $x_0 \in G_0$, just take the first sum in (2.1) as $\sum_{k=0}^{N}$, and the Gibbs derivative has the form:

$$\begin{aligned} f^{<1>}(x) &= \lim_{N \to +\infty} \Delta_N f(x) \\ &= \lim_{N \to +\infty} \sum_{k=0}^{N} p^k \sum_{j=0}^{p-1} A_j f(x \oplus j p^{-k-1}), \quad x \in G_0. \end{aligned}$$

In 1983, Z.L.He [10] generalized Gibbs derivatives to the p_j-adic groups in fractional forms for $j \in \mathbf{P}$ in $L^r(G_0)$ strong sense

$$D^{<\alpha>} * f(x) = [\sum_{k=0}^{p_j-1} k^\alpha \overline{\psi_k}] * f(x), \quad x \in G_0, \tag{2.2}$$

where $\{\psi_k(x)\}_{k=0}^{+\infty}$ is p_j-adic Walsh system.

In 1988, He [24] generalized Gibbs derivatives again to the p_j-adic groups in fractional forms for $j \in \mathbf{P}$ in $L^r(G)$ strong sense for the locally compact case $G = \mathbf{R}, \mathbf{T}, \mathbf{Z}$, $1 < r < +\infty$.

From 1981 to 1990, we completed the following:

1 Generalizations of the dyadic Gibbs derivatives to that of p-adic [1] case and p_j-adic [11], [25] cases;

2 Proved important properties of Gibbs derivatives, including

   (a) Operation properties [1]-[5],[7],[10],[11],[24];

   (b) Fourier-Walsh transform properties (also distribution sense) [5],[7],[23],[28];

   (c) Approximation properties (Jackson and Bernstein theorems) [5],[8]-[10],[12],[24],[28];

   (d) Construction of some approximation identity kernels, such as, Abel-Poisson type kernels [6],[20], a class of approximation identity kernels [13]-[15], Vallee-Poussin kernels [30], and so on [39].

Gibbs derivatives on local fields

Since the dyadic analysis is a special one of p-adic case, which is a special case of local fields, so we have paid our attention to the study of analysis on local fields from 1984.

Let K be a locally compact, totally disconnected, complete topological field with non-archimedean norm, $x \to |x|$ a mapping from K to $\mathbf{R}^+$, such that:

$$\begin{array}{rc} (i) & |x| \geq 0, |x| = 0 \iff x = 0; \\ (ii) & |xy| = |x||y|; \\ (iii) & |x+y| \leq \max\{|x|, |y|\}. \end{array}$$

Thus it has a topological base $\tau = \{G_k\}_{k\in\mathbf{Z}}$ of $0 \in K$ as

$$G_k = \{x \in K : |x| \le q^{-k}\}, \quad k \in \mathbf{Z},$$

with

(a) $\{G_k\}_{k\in\mathbf{Z}}$ is strictly decreasing, $\cdots \subset G_{k+1} \subset G_k \subset G_{k-1} \subset \cdots$, $k \in \mathbf{Z}$, every G_k is closed, open and compact set;

(b) $\bigcup_{k=-\infty}^{+\infty} G_k = K$, and $\bigcap_{k=-\infty}^{+\infty} G_k = \{0\}$, and $q = p^c$, $c \in \mathbf{N}$, p is a prime.

Moreover, let Γ be the character group of K, then the annihilators of G_k is

$$\Gamma_k = \{\xi \in \Gamma : |\xi| \le p^k\}, \quad k \in \mathbf{Z}$$

with

(a') $\{\Gamma_k\}_{k\in\mathbf{Z}}$ is strictly increasing, $\cdots \subset \Gamma_{k-1} \subset \Gamma_k \subset \Gamma_{k+1} \subset \cdots$, $k \in \mathbf{Z}$, every Γ_k is closed, open and compact set;

(b') $\bigcup_{k=-\infty}^{+\infty} \Gamma_k = \Gamma$, and $\bigcap_{k=-\infty}^{+\infty} \Gamma_k = \{0\}$.

In 1985, [18], W.X. Zheng generalized the Gibbs derivatives to local fields. He still uses the limit to define the Gibbs derivatives on K:

$$\begin{aligned} f^{<1>}(x) &= \lim_{N\to+\infty} \Delta_N f(x) \qquad (2.3) \\ &= \lim_{N\to+\infty} \sum_{l=-N+t}^{N+t} q^{-n-j+1} \sum_{l=0}^{q^N-1} \sum_{v=0}^{p-1} \exp\left(\frac{-2\pi i}{p}\right) f(x + l\beta^{-j}) \end{aligned}$$

for a fixed $t \in \mathbf{N}$, $q = p^c$, $c \in \mathbf{N}$, p is a prime. Later, he gave some important properties in [15]-[17], [19], [21], [22], [25]-[27], [31]-[34].

In 1993, [38], H.K. Jiang generalized again the Gibbs derivatives to a-adic group for

$$a = \{\cdots, a_{-n}, \cdots, a_{-1}, a_0, a_1, \cdots, a_n, \cdots\}, \quad a_j \ge 1$$

by also a limit form.

Then, in 1992, [35], W.Y. Su gave the definitions of Gibbs derivatives and Gibbs integrals by the so called pseudo-differential operators:

For a Haar measurable function $f : K \to \mathbf{C}$, if the integral

$$T_{<\cdot>^m} f(x) = \int_\Gamma \{ \int_K < \xi >^m f(t) \overline{\chi_\xi}(t-x) dt \} d\xi, \quad m \ge 0 \qquad (2.4)$$

exists at $x \in K$ with $< \xi >= \max\{1, |\xi|\}$, then $T_{<\cdot>^m} f(x)$ is called the pointwise Gibbs derivative of f at $x \in K$ with order m, denoted by

$$f^{<m>}(x) = T_{<\cdot>^m} f(x), \quad m \ge 0. \qquad (2.5)$$

Also we can define the $L^r(K)$-strong Gibbs derivatives with order m.

And for $m \leq 0$, it is the pointwise Gibbs integral of f at $x \in K$ with order m, denoted by

$$f_{<m>}(x) = T_{<\cdot>^m} f(x), \quad m \leq 0. \tag{2.6}$$

Many important properties of Gibbs derivative $f^{<m>}(x) = T_{<\cdot>^m} f(x), m \geq 0$ and Gibbs integral $f_{<m>}(x) = T_{<\cdot>^m} f(x), m \leq 0$ are shown in [35], [36], [40]-[43], including the operation properties, approximation operators and approximation properties, all using the Gibbs derivatives.

2. Function Spaces

Not only in "harmonic analysis", "differential equations", but also in the "fractal analysis", lots of function spaces play very important roles, so we have to establish some function spaces and study properties by using Gibbs derivatives.

Besov type spaces, Triebel B-type spaces and F-type spaces

In 1988, W.Y. Su studied the boundenss of the pseudo-differential operators in Besov type spaces on a local field [26]. As special cases, she got Lebesgue type spaces, Sobolev spaces in [26].

In 1989, C.W.Onneweer and W.Y.Su defined the homogeneous Besov spaces on the Vilenkin groups, studied some properties about dual spaces and got many interesting and useful results [29].

In 1992, G.C. Zhou and W.Y. Su defined the Triebel B-type spaces $B^s_{p,q}(K_n)$ and F-type spaces $F^s_{p,q}(K_n)$ on n-dimension local field K_n, proved some embedding theorems, and lifting properties [37].

Holder type spaces $C^\alpha(K)$

In 2006, [46], W.Y. Su gave the definition of the Holder spaces, and proved a very interesting property: the Holder spaces are exactly describe the Gibbs smoothness.

In fact, the Holder type space $C^\sigma(K)$, $\sigma \in [0, \infty)$ is defined just by the Littlewood-Paley decomposition [46], and we proved that

THEOREM 2.1 *The Holder type space $C^\sigma(K)$, $\sigma \in [0, \infty)$ has the following properties:*

(1) if $f \in C^\sigma(K)$, then for $\forall 0 \leq \lambda \leq \sigma$, function f has the Gibbs-type derivative $T_{<.>^\lambda} f(x)$, $x \in K$, and $T_{<.>^\lambda} f \in C^{\sigma-\lambda}(K)$;

(2) if $T_{<.>^\sigma} f \in C^0(K)$, then for $\forall 0 \leq \lambda \leq \sigma$, function f has the Gibbs-type derivative $T_{<.>^{\sigma-\lambda}} f(x)$, $x \in K$, and $T_{<.>^{\sigma-\lambda}} f \in C^\lambda(K)$.

Lipschitz classes $Lip(\alpha, K)$

In 2007, "Lipschitz classes on local fields" has published, some essential properties are included in it [49].

For $0 < \alpha < +\infty$, we call the function class

$$Lip(\alpha, K) = \{f \in C(K) :\| f(\cdot + h) - f(\cdot) \|_{C(K)} = O(|h|^{\alpha})\}$$

the Lipschitz class on local field K. And the relationship between the Holder type spaces and the Lip classes are also reviled:

THEOREM 2.2 *In a local field K, we have*

$$Lip(\alpha, K) = C^{\alpha}(K), \quad \alpha \in (0, +\infty).$$

3. Comparison Between the Classical Derivatives and Gibbs Derivatives

In the case of the Euclidean space $\mathbf{R}^n$ as an underline space, we use

$$C^m \equiv C^m(\mathbf{R}^n), \quad m \in \mathbf{P}$$

to denote the function space of m-order continuous differential functions. To describe the smoothness of functions defined on $\mathbf{R}$, we list as follows [49]

$$\cdots \not\subseteq C^{m+1} \not\subseteq C^m \not\subseteq C^1 \not\subseteq Lip1 \not\subseteq Lip\alpha \not\subseteq Lip\beta \not\subseteq \cdots C$$

$$m \in \{2, 3, \cdots\} \qquad\qquad 1 > \alpha > \beta > 0$$

(2.7)

We note that, there is a "gap" between $Lip\alpha$ and $Lip^*\alpha$; moreover, also has a "gap" for the Holder space C^{α}, $\alpha \in (0, +\infty)\backslash\mathbf{N}$, and Zygmund space C^{α}_*, $\alpha \in (0, +\infty)\backslash\mathbf{N}$:

$$0 < \alpha < 1 : C^{\alpha} \leftrightarrow Lip\alpha = Lip^*\alpha \leftrightarrow C^{\alpha}_*,$$

$$\alpha = 1 \qquad\qquad Lip1 \not\subseteq Lip^*1$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \text{no Holder space} & & \text{no Zygmund space} \end{array}$$

$$1 < \alpha < 2: \quad C^\alpha, \qquad\qquad Lip^*\alpha \leftrightarrow C_*^\alpha.$$

$$\begin{array}{c} \uparrow \\ \text{no } Lip \text{ class} \end{array}$$

However, in the case for that of local field K as an underline space, "p-type smoothness" is described by Holder type space $C^\sigma(K)$, $\sigma \geq 0$ fully by Theorem 1 in [46], no needed 2-order difference:

$$C^\alpha(K) \leftrightarrow Lip(\alpha, K), \quad \alpha \in (0, +\infty). \tag{2.8}$$

Moreover, no "gap", there is a corresponding nice relationship

$$\begin{array}{ccc} C^\alpha(K) & \subset & C^\beta(K), \quad 0 \leq \beta < \alpha < +\infty \\ \updownarrow & & \updownarrow \\ Lip\alpha & \subset & Lip\beta \end{array}$$

and it shows that the space with more higher smoothness is contained in that of the lower one.

We see the essential differences between the analysis of $\mathbf{R}$ and K.

(1) the algebraic structure of $(\mathbf{R}, +, \times)$ and $(K, \oplus, \otimes)$ are totally different;

(2) the topological structures of $(\mathbf{R}, +, \times, \tau)$ and $(K, \oplus, \otimes, |\cdot|)$ are totally different: $\mathbf{R}$ is connected, and K is totally disconnected;

(3) the structures of the corresponding character groups of $\mathbf{R}$ and K are totally different: by the dual theory, $\Gamma_{\mathbf{R}} \leftrightarrow \mathbf{R}(\Gamma_{[-1,1]} \leftrightarrow \mathbf{T})$, $\Gamma_K \leftrightarrow K(\Gamma_{G_0} \leftrightarrow \{0\} \bigcup \mathbf{N})$, thus $\Gamma_{\mathbf{R}}$ and Γ_K are connected and totally disconnected, respectively;

(4) the character equations (eigen-equations) of $\mathbf{R}$ and K are totally different: they are $y' = \lambda y$ and $y^{<1>} = \lambda y$ respectively, where y' is the classical derivative of y on $\mathbf{R}$, which is "the rate of change at $x \in \mathbf{R}$", and $y^{<1>}$ is the Gibbs derivative of y on K, which describes "a global rate of change on K", respectively; and the non-zero values λ(eigen-values), which make character equations have non-zero solutions, are $\lambda = iy$ and $\lambda = y$, respectively; and the non-zero solutions are character functions $\exp 2\pi iyx$ and $\exp \frac{2\pi iy \odot x}{p}$, respectively;

(5) the best approximation equivalent theorems on $\mathbf{R}$ and K are totally different: in fact, we have for Euclidean space as an underline space

$$\begin{aligned} 0<\alpha<1: \quad & f\in Lip\alpha \Leftrightarrow E_n(f,C_{[0,2\pi]})=O(n^{-\alpha}); \\ \alpha=1: \quad & f\in Lip1 \Rightarrow E_n(f,C_{[0,2\pi]})=O(n^{-1}), \\ & f\in Lip^{*}1 \Leftrightarrow E_n(f,C_{[0,2\pi]})=O(n^{-1}); \end{aligned}$$

where $E_n(f,C_{[0,2\pi]})$ is the best approximation of function $f\in C_{[0,2\pi]}$. However, for the case of local fields, on the compact subgroup $G_0=D=\{x\in K:|x|\le 1\}$, the following 4 terms are equivalent for $\alpha>0$, $r\in\{0\}\bigcup\mathbf{N}$

$$\begin{aligned} (i) \quad & f^{<r>}\in Lip(\alpha,G_0); \\ (ii) \quad & \omega(f^{<r>},p^{-n},C(G_0))=O(p^{-n\alpha}), \quad n\to\infty; \\ (iii) \quad & E_{p^n}(f,C(G_0))=O(p^{-n(\alpha+\gamma)}), \quad n\to\infty; \\ (iv) \quad & \parallel f(\cdot)-S_{p^n}(f,\cdot)\parallel_{C(G_0)}=O(p^{-n(\alpha+\gamma)}), \quad n\to\infty. \end{aligned}$$

The similar results for $K^{+}=K$ hold for the local field K [3]: for α and r, $\alpha>0$, $r\in\{0\}\bigcup\mathbf{N}$, the following statements are equivalent

$$\begin{aligned} (i) \quad & f^{<r>}\in Lip(\alpha,K); \\ (ii) \quad & \omega(f^{<r>},p^{-n},C(K))=O(p^{-n\alpha}), \quad n\to\infty; \\ (iii) \quad & E_{p^n}(f,C(K))=O(p^{-n(\alpha+\gamma)}), \quad n\to\infty. \end{aligned}$$

Thus, the 1-order continuity modulus determines the equivalent theorems on a local field K, so does $Lip\alpha$ class.

We have analyzed the properties of the group operations, topological structures, character groups, rate of changes (thus, the classical derivatives and Gibbs derivatives, as well as eigen-equations and eigen-values), and the approximation structures of $\mathbf{R}$ and K. So we can see that the formulas (2.7) and (2.8) describe the smoothness of functions defined on $\mathbf{R}$ and K, respectively, and we also may understand that: "classical derivatives" is suitable for the analysis on $\mathbf{R}$ and "Gibbs smoothness" is suitable for the analysis on local fields.

Moreover, we need introduce Lip^{*} class on $\mathbf{R}^n$ since the equivalent theorems need 2-order continuity modulus, and just need Lip class on K, and have $C^{\alpha}(K)\leftrightarrow Lip\ (\alpha,K)$, since we just need 1-order continuity modulus.

Recall the Theorem 1.1 in [13]: If $f\in L^1$, then for each $t\in(-\infty,+\infty)$, it holds

$$\lim_{p\to\infty}\int_0^{+\infty} f(x)\overline{\omega}_p(t,x)dx=\int_0^{+\infty} f(x)\exp[-2\pi i(tx-(sngt)\{|t|\}\{x\})]dx,$$

where $\{x\}$ is the decimal part of a real number x, $\omega_p(t,x)$ is the p-series Walsh function. This theorem not only shows that the essential difference between the analysis on the Euclidean spaces and local fields, but also proves that: kinds of results of the Fourier analysis on local fields, are never the special cases of those on Euclidean spaces when $p \to \infty$, thus the guesswork is denied: results in Fourier analysis over local fields are just special cases of those over Euclidean spaces.

Moreover, we may connect this fact with the famous result in fractal geometry: increase the numbers of sides of the Koch curve, the angle $60°$ can never be disappeared.

Thus, we may interpreter that: $\mathbf{R}^n$ analysis is a powerful tool to describe the universe in the macroscopic point of view, and the analysis over local fields is suitable to serve to the point of view in microcosmic spaces.

4. Principles for Defining "rate of change"

In the classical case, Newton's derivatives have a meaning "rate of change", so we may ask: does it have the similar meaning for the Gibbs derivatives? In other words, does it have some principles for defining operations which have the meaning "rate of change"? It seems to have as follows.

(1) Differentiation has an inverse operator - the integral operator

$$\frac{d}{dx}\int f(x)dx = f(x), \quad \int df(x) = f(x);$$

(2) Derivatives have Fourier transform formula - Fourier transform of $f'(x)$

$$[f'(\cdot)]^{\wedge}(\xi) = i\xi f^{\wedge}(\xi);$$

(3) Differentiation has approximation properties - direct and inverse theorems, such as

$$E_n(f, L_r(R)) = O(n^{-\alpha-s}) \iff f^{<s>} \in Lip(\alpha, L_r(R));$$

(4) Differentiation satisfies eigen-equations - eigen-functions, eigen-values

eigen-equations	$\frac{dy}{dx} = \lambda y;$
eigen-values	$\lambda = i\xi; \quad \xi \in \mathbf{R};$
eigen-functions	$e^{i\xi x}, x \in \mathbf{R}, \quad \xi \in \mathbf{R}$

(5) Function spaces which differential functions live —- spaces on $\mathbf{R}$ shch as $C^n, C^\alpha, C_*^\alpha$.

Then for Gibbs derivatives, the above five essential properties all have been proved (see [35]). Thus we may conclude that: Gibbs derivatives make sense for describing the rate of change for lots of natural phenomena, and are certainly very suitable for scientific studies in many scientific fields.

5. Applications to Fractal Analysis by Gibbs Derivatives

Fractals, no classical derivatives, such as the Wererstrass function, it is a typical example nowhere has derivative, and it is also a typical example in fractal analysis. Other examples of fractal: The Brownean motions, Cantor set, Julia set, Kock curve, and so on.

How to describe the rate of change of a fractal?

We get more and more idea that Gibbs derivatives are nice tools to describe the rete of changes for these functions which have no classical derivatives.

Recently, we have some papers to study the p-adic Gibbs derivatives on local fields, such as the Cantor functions, the Weierstress-like functions, Weierstress type functions, and so on, see [47],[48],[50]-[54].

In [47], H. Qiu, W.Y. Su show that for the Weierstrass-like function on dyadic local field K_2 as

$$g(x) = \begin{cases} \sum_{j=1}^{+\infty} x_j (\frac{1}{2})^j, & \forall x \in B_1, \\ 0, & \text{otherwise} \end{cases}$$

where $B_1 = \{x \in K_2 : |x| \leq 2^{-1}\}$, supp$g \subset B_1$, the prime element $\beta \in K_2$ with non- archimedean norm $|\beta| = 2^{-1}, x_j \in \{0, 1\}, j = 1, 2, \cdots$.

THEOREM 2.3 *The function $g(x)$ is infinitely integrable; and is an m-order differentiable with $0 < m < 1$, and*

$$g^{<m>}(x) = \begin{cases} \frac{1}{4} + \frac{2^m}{4} - \frac{2^{2m-2}}{1-2^{m-1}} + \sum_{j=1}^{+\infty} x_j (\frac{1}{2})^{j-(j+1)m}, & |x| \leq 2^{-1}, \\ \frac{1}{4} - \frac{2^m}{4}, & |x| = 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover, there is no 1-order derivatives at any point in B_1.

In [52], we consider the 3-adic Cantor function $f(x)$ on 3-series local field K_3 with

$$x \in K_3 \Rightarrow x = \sum_{j=-s}^{+\infty} x_j \beta^j, \quad x_j \in \{0, 1, 2\}, j = -s, -s+1, \cdots, s \in \mathbf{Z},$$

with $|\beta| = 3^{-1}$.

And we define 3-adic Cantor function as:

$$f(x) = \begin{cases} \sum_{j=0}^{k-2} (x_j - 1) \cdot (\frac{1}{2})^{j+1} + (\frac{1}{2})^k, & x \in D, x_{k-1} = 0; \\ & x_j \neq 0, 0 \leq j \leq k-2; \\ \sum_{j=0}^{+\infty} (x_j - 1) \cdot (\frac{1}{2})^{j+1}, & x \in D, x_j \neq 0, 0 \leq j < +\infty; \\ 0, & x \notin D, \end{cases}$$

where $D = \{x \in K_3 : |x| \leq 1\}$.

Then, we have

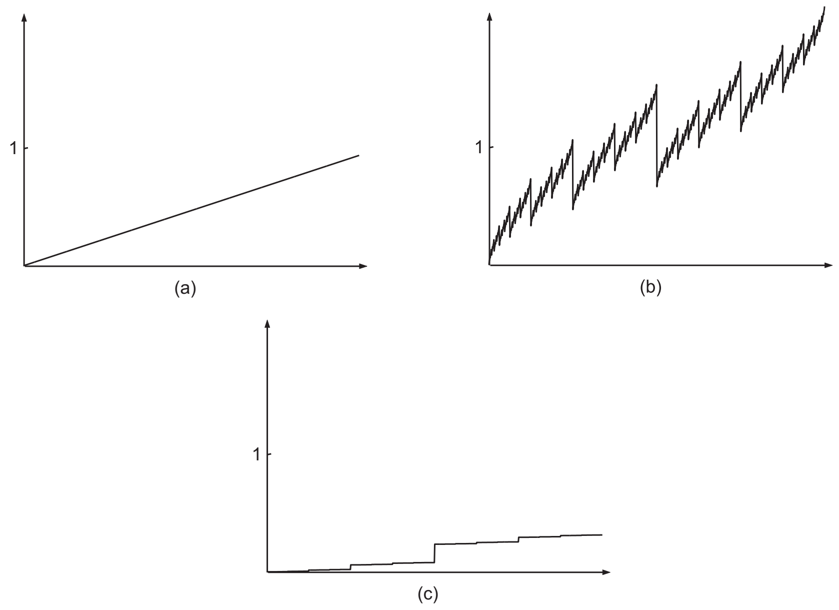

Figure 2.1. The sketch maps of: (a) g on K_2; (b) the $1/2$-order derivative of g on K_2; (c) the 1-order integral of g on K_2.

THEOREM 2.4 *$f(x)$ is infinitely integrable; and is an m-order differentiable with $0 \leq m < \frac{\ln 2}{\ln 3}$, and for some character χ of K, for $x \in D$,*

$$\begin{aligned} f^{<m>}(x) &= \frac{1}{2} + \sum_{l=1}^{+\infty} \frac{3^{lm}}{6^l} \chi(\beta^{-l}x)((\frac{1}{2} + \omega) + (\frac{1}{2} + \overline{\omega})\chi(\beta^{-l}x)) \\ &\quad \times \prod_{j=1}^{l-1} (2 - \chi(\beta^{-j}x) - \chi^2(\beta^{-j}x)), \end{aligned}$$

for $x \notin D$, $f^{<m>}(x) = 0$. Moreover, the Hausdorff dimension of the image of $f^{<m>}(x)$ with domain D is always 1 for all $-\infty < m \leq \frac{\ln 2}{\ln 3}$.

In [53], we study the Weierstrass type function

$$W(x) = \sum_{k=1}^{+\infty} p^{(s-2)k} \mathrm{Re}\chi(\beta^{-k}x),$$

where $x \in D$, $1 \leq s < 2$, in p-series field.

Then, we have

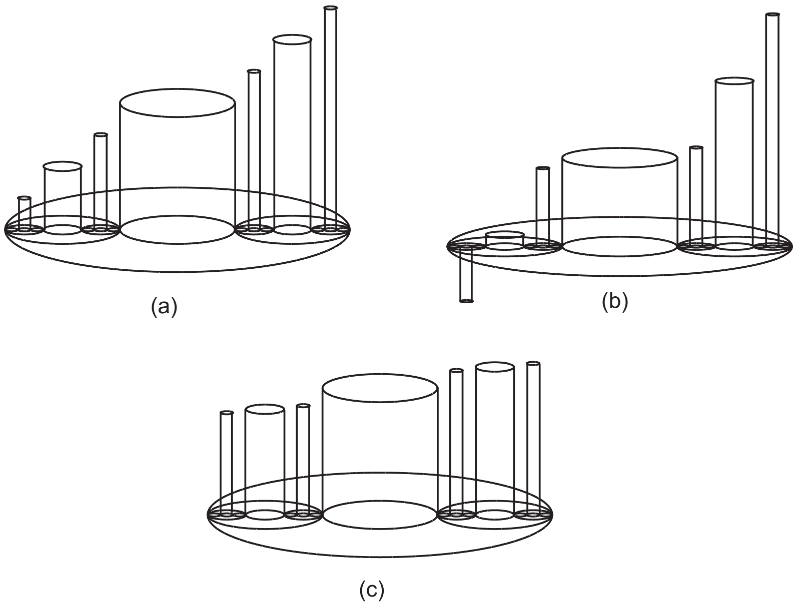

Figure 2.2. The sketch map of: (a) 3-adic Cantor function on K_3; (b) $1/2$-order derivative of the 3-adic Cantor function; (c) 1-order integral of the 3-adic Cantor function.

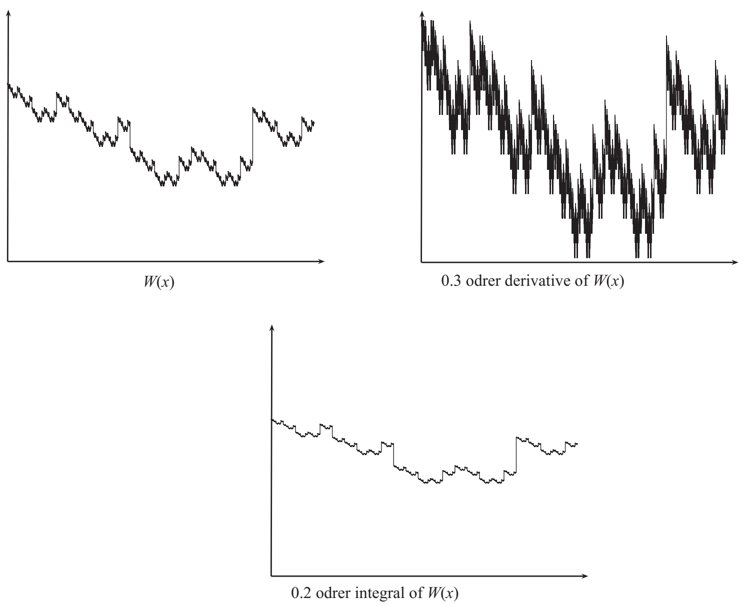

Figure 2.3. The sketch maps of $W(x)$, $W^{<0.3>}(x)$ and $W_{<0.2>}(x)$ for $p = 5$, $s = 1.45$.

THEOREM 2.5 *The function $W(x)$ is infinitely integral and m-order differentiable with $m < 2 - s$, and*

(1) For the Box dimension dim_B and Paking dimension dim_P, we have

$$dim_B \Gamma(W(x)^{<m>}, D) = dim_P \Gamma(W(x)^{<m>}, D) = s+m, \quad m \in [1-s, 2-s);$$

(2) For Hausdorff dimension, we have

$$dim_H \Gamma(W(x)^{<m>}, D) = s + m$$

where

$$\begin{array}{ll} m \in (1-s, 2-s), & \text{if } p = 2, \\ m \in (\log_p(2p-1) - s, 2 + \log_p y(b_p) - s), & \text{if } p > 2. \end{array}$$

This theorem is very interesting and important, since it gives connected relationship between the Gibbs derivatives and fractal dimensions.

More applications of Gibbs derivatives are to medical study, for example, the study of liver's cancer, gene's action in liver's cancer, and so on.

We will continue our study on Gibbs derivatives, in theory and applications.

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Chapter 3

CONSTRUCTION OF DISSIPATIVE DYNAMICAL SYSTEM IN THE SENSE OF PRIGOGINE USING GIBBS DERIVATIVES

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Abstract The paper elaborates construction of dissipative dynamical systems in the sense of Prigogoine by using theory of Walsh harmonic analysis and dyadic differential calculus.

1. Introduction

Prigogine and his research group investigated in several books and papers the possibility for the construction of highly-unstable non-conservative discrete time deterministic dynamical systems which model locally by its states and state transition microscopic mechanism and generate globally complex state patterns, such as chaotic attractors, which have relevance to observed behaviours of complex systems as they appear in different fields of science. Such an approach should supplement the existing methods, based on stochastic processes and statistical mechanics methods.

In this paper we follow Prigogine in such a construction and elaborate an example which is given in the books [7] and [8]. However, we use in our example. the existing theory of Walsh harmonic analysis and the theory of dyadic derivates as introduced by Edmund Gibbs [2]. The Λ-transform of Prigogine is identified by us as a dyadic convolution operation and is introduced by the dyadic derivative operator G. The topic which we discuss here was already followed by the author in an earlier report [5] and independently by the paper [1]. In [6] a similar example using as state space the Hilbert space of square integrable real functions was given. The topic of this paper here was also covered in the lecture "On the role of Walsh Functions for certain dynamical systems" at the workshop *Dyadic Analysis with Applications and Generalizations*, 2003 June 11-13, Balatonszemes, Hungary, organized by Ferenc Schipp.

2. Baker Transform and its Dyadic Representation

Let E denote the unit square $E : [0, 1) \times [0, 1)$. The *baker transform* B can be defined by the mapping

$$B(x, y) := \begin{cases} (2x, \frac{1}{2}y), & \text{if } 0 \leq x < \frac{1}{2}, \\ \\ (2x - 1, \frac{1}{2}(y + 1)), & \text{for } \frac{1}{2} \leq x < 1, \end{cases} \tag{3.1}$$

where $(x, y) \in E$.

The baker transform B is invertible. For its inverse B^{-1} we get

$$B(x, y) := \begin{cases} (\frac{1}{2}x, 2y), & \text{if } 0 \leq y < \frac{1}{2}, \\ \\ (\frac{1}{2}x + 1, 2y - 1), & \text{for } \frac{1}{2} \leq x < 1. \end{cases} \tag{3.2}$$

The baker transform moves the points in the following manner: The left half of E is folded to its half size and stretched to full length such that it becomes the lower half of E. The right half of E becomes by B the upper half of E. Iteration of B mixes the points of E dramatically such that two points (x, y) and (x', y') which are close to each other get a far distance.

The operation of B, however, becomes very simple if we represent E by its associated *dyadic group* D. To show this let denote by $x(i)$, $i = 1, 2, \ldots$ and by $y(j)$, $j = 1, 2, \ldots$ the coefficients of the dyadic representation of x and y, respectively. Then the infinite vectors $(x(1), x(2), \ldots)$ and $(y(1), y(2), \ldots)$ can be concatenated to form the both-sided infinite vector

$$(\ldots y(2), y(1); x(1), x(2), \ldots),$$

which is the associated element of (x, y) in the dyadic group D. It is straightforward to see, that the baker transform B degenerates on D to a simple shift-operation.

The value $B(x, y)$ is associated in D to $(\ldots y(2); y(1), x(1), x(2), \ldots)$ which is derived by 1-step right shift of $(\ldots y(2), y(1); x(1), x(2), \ldots)$. It is evident, that by this property of B with respect to its representation on the dyadic group D the Walsh functions, being group characters of D will have a specific role in the analysis of B.

3. Baker-dynamical Systems

We use in the following the baker transform, to define the state transition function for a autonomous discrete time dynamical system, which we will call baker-dynamical system.

Let $L_2(E)$ denote the Hilbert space of square-integrable real functions defined on E. Let B^* denote the iteration of E of arbitrary length. Then, the pair $(L_2(E), B^*)$ is called *a baker dynamical system.*

The space $L_2(E)$ constitutes *the state space* and B^* *the (global) state transition function.* B is *the next state transition function.*

Since B is measure preserving and also invertible, a baker dynamical system is *conservative* and *reversible*, that means that we have for all states the following equations

$$\|B^*(q)\|^2 = \|q\|^2 \quad \text{and} \quad (B^{-1})^* B^*(q) = q. \tag{3.3}$$

4. Walsh-Fourier analysis of Baker-dynamical Systems

Let $\psi(n)$, $n = 0, 1, 2, \ldots$ denote *the Walsh-Paley functions.* The 2-D Walsh functions $\psi(h, v)$, $h, v = 0, 1, 2 \ldots$ are defined by the product

$$\psi(h, v) := \psi(h)\psi(v), \quad h, v = 0, 1, 2, \ldots \tag{3.4}$$

Each Walsh function $\psi(h, v)$ is a real function on E with values $\psi(h, v)(x, y) = \psi(h)(x)\psi(v)(y)$ with $(x, y) \in E$.

The system $\{\psi(h, v) : h, v = 0, 1, 2, \ldots\}$ of 2-D Walsh functions is known to constitute a complete orthonormal function system for the Hilbert space $L_2(E)$. Each Walsh function can therefore be considered also as a state of the baker-dynamical system $(L_2(E), B^*)$ and each state q can be represented by a Walsh-Fourier series of the form

$$q = \sum_{h,v} \hat{q}(h, v)\psi(h, v). \tag{3.5}$$

By using (3.5) the next state $B(q)$ can be computed by

$$B(q) = \sum_{h,v} \hat{q}(h, v)B\psi(h, v). \tag{3.6}$$

It has been shown in [5] that the set of Walsh functions $\psi(h, v)$ is closed under the baker transform B and we have

$$B\psi(h, v) = \begin{cases} \psi(2h, \frac{1}{2}v), & \text{if } v \text{ is even,} \\ \psi(2h + 1, \frac{1}{2}(v - 1)), & \text{if } v \text{ is odd.} \end{cases} \tag{3.7}$$

By the result of (3.7) we can compute $B(q)$ of (3.6) now as

$$\begin{aligned} B(q) \quad = \quad & \sum_{u,v(even)} \hat{q}(h, v)\psi(2h, v/2) \\ & + \sum_{u,v(odd)} \hat{q}(h, v)\psi(2h + 1, (v - 1)/2). \end{aligned} \tag{3.8}$$

By substituting $h^* := 2h$ and $v^* := v/2$ if v is even, and $h^* := 2h+1$ and $v^* := (v-1)/2$ if v is odd we are able to write (3.8) as

$$\begin{aligned} B(q) &= \sum_{u,v(even)} \hat{q}(h/2, 2v)\psi(h,v) \\ &\quad + \sum_{u,v(odd)} \hat{q}((h-1)/2, 2v+1)\psi(h,v). \end{aligned} \tag{3.9}$$

If we define the function $\hat{B}$ on the Hilbert space $L_2(\mathbf{N}_0 \times \mathbf{N}_0)$, where $\mathbf{N}_0$ denotes the set of numbers $0, 1, 2, \ldots$ by

$$\hat{B}(\hat{q}) = \begin{cases} \hat{q}(h/2, 2v), & \text{if } h \text{ is even,} \\ \\ \hat{q}((h-1)/2, 2v+1), & \text{if } v \text{ is odd.} \end{cases} \tag{3.10}$$

we are able to write (3.9) as

$$B(q) = \sum_{h,v} \hat{B}(\hat{q} = \psi(h,v). \tag{3.11}$$

The operation $\hat{B}$ on $L_2(\mathbf{N}_0 \times \mathbf{N}_0)$, which is given by

$$\hat{B}(h,v) = \begin{cases} (h/2, 2v), & \text{if } h \text{ is even,} \\ \\ ((h-1)/2, 2v+1), & \text{if } v \text{ is odd,} \end{cases} \tag{3.12}$$

can be called *the discrete baker transform.*

The result (3.11) shows that whenever a state q is mapped by the baker transform B to a next state $B(q)$, the related Walsh-Fourier spectrum $\hat{q}$ of q is mapped by the discrete baker transform $\hat{B}$ to $\hat{B}(\hat{q})$.

There are several invariants to observe in a baker-dynamical system $(L_2(E), B^*)$.

First it is to observe by (3.8) that the baker transform preserves for all states q of $(L_2(E), B^*)$ the multiplicity $V(q)$ as defined by [3]. In addition by the validity of the Parseval theorem we have

$$\|B(q)\|^2 = \|q\|^2 = \|\hat{q}\|^2 = \|\hat{B}(\hat{q})\|^2 \tag{3.13}$$

5. Construction of a Λ - transform in the sense of Prigogine

In the following we construct by means of the dyadic derivative as introduced by Edmund Gibbs a specific transform for the states q of $(L_2(E), B^*)$ which is in line with the concept of the Λ-transform as introduced by the work of Prigogine. We follow here the presentation which is given in the Chapter V of the book [8].

A basic concept is there the T-operator of Misra [4] which can be defined in our context as an operator which has the Walsh functions $\psi(h, v)$ as its eigenfunctions

$$T\psi(h, v) := \tau(h, v)\psi(h, v). \tag{3.14}$$

Each eigenvalue $\tau(h, v)$ can be called according to Misra the "dyadic time" of $\psi(h, v)$. From this definition we can conclude that a T-operator is a dyadic convolution operator. There are of course many ways to define convolution operators. One of it, which seems to be in line with the original ideas of Misra can be constructed by taking $\tau(h, v) := h + v$. To meet this case we can define T by the use of the following *(strong) Gibbs differential operator G* for functions of $f \in L_2(E)$ which can be defined by

$$Gf(x, y) := \lim_{(n\to\infty)} G(n)/dxf(x, y) + \lim_{(n\to\infty)} G(n)/dyf(x, y), \tag{3.15}$$

where the operations $G(n)/dx$ and $G(n)/dy$ are defined by

$$G(n)/dx := \sum_{i=0}^{n-1} 2^{i-1}(f(x, y) - f(x \oplus 2^{-i-1}, y)) \tag{3.16}$$

and by

$$G(n)/dy := \sum_{j=0}^{n-1} 2^{j-1}(f(x, y) - f(x, y \oplus 2^{-j-1})). \tag{3.17}$$

It can be shown that G as defined by (3.15) meets the desired requirements for a T-operator and we have

$$G\psi(h, v) = (h + v)\psi(h, v). \tag{3.18}$$

In addition G fulfills with respect to the baker transform the requirement of non-commutativity, that means that

$$GB \neq BG. \tag{3.19}$$

This is most easy to show for taking Walsh functions as arguments. For $GB\psi(h, v)$ we get $(h^* + v^*)\psi(h^*, v^*)$ with h^* and v^* given according to (3.7), for $BG\psi(h, v)$ however we have $(h + v)\psi(h^*, v^*)$. Since $h^* + v^* \neq h + v$ we conclude that (3.19) holds.

The next step is to make use of G to construct a Λ-transform. Prigogine shows in his work (see for example [8]) that Λ can be constructed by the Misra T-operator in the following way:

Chose the Walsh-Fourier spectrum $\hat{\Lambda}$ of Λ such that it is a monotonic decreasing function of $\hat{T}$. In our case, since we have $\hat{T} = \hat{G}$ which is given by $\hat{G}(h, v) = h + v$ a Λ-operator can be defined by $\hat{\Lambda}(h, v) := 1/(h + v)$. However, this is exactly the inversion of the Gibbs differential operator G in the spectral domain, which means that we can consider in our case the Λ-operator as being equivalent to the dyadic integral operator σ which is the inverse of G.

6. Construction of a Baker-Prigogine Dynamical System by Gibbs Differentiation

Following again [8] the dynamical system $(L_2(E), \Gamma^*)$ is defined by

$$\Gamma := \Lambda B. \tag{3.20}$$

We call $(L_2(E), \Gamma^*)$ *a Gibbs-baker-dynamical system*, for short a GbP-dynamical system.

Our goal is to show in the following, that a GbP-dynamical system meets the requirements of being a dissipative dynamical system. This means that it has a non-conservative dynamics in the sense that the realization of its state trajectories consume energy and that in addition the global state transition function Γ^* performs a strong mixing operation on the states. This is, however guaranteed by the baker transform B as a factor of Γ. To prove that Γ^* generates a non-conservative dynamics it is sufficient to show that for all states $q \in L_2(E)$ the following inequality holds

$$\|\Gamma(q)\|^2 < \|q\|^2. \tag{3.21}$$

By the Parseval equation the inequality (3.21) is equivalent to the inequality

$$\|\hat{\Gamma}(\hat{q})\|^2 < \|\hat{q}\|^2. \tag{3.22}$$

Computation of the left side of (3.22) gives

$$\begin{aligned}
\hat{\Gamma}(\hat{q}(h, v)) &= \hat{\Lambda}\hat{B}\hat{q}(h, v) = \hat{\Lambda}\hat{q}(\hat{B}(h, v)) \\
&= \begin{cases} \hat{\Lambda}\hat{q}(h/2, 2v), & \text{if } h \text{ is even,} \\ \\ \hat{\Lambda}(2h + 1, (v - 1)/2)\hat{q}(h, v), & \text{if } v \text{ is odd,} \end{cases}
\end{aligned}$$

by substituting $h^* := h/2$ and $v^* := 2v$ we can write

$$\hat{\Gamma}(\hat{q}(h, v)) = \begin{cases} \hat{\Lambda}(2h, v/2)\hat{q}(h, v), & \text{if } v \text{ is even,} \\ \\ \hat{\Lambda}(2h + 1, (v - 1)/2)\hat{q}(h, v), & \text{if } v \text{ is odd.} \end{cases} \tag{3.23}$$

The result of (3.23) allows us to evaluate the correctness of the inequality (3.22). We have to distinguish the following two cases:

Case 1: v is even

In this case we have

$$\begin{aligned}\|\hat{\Gamma}(\hat{q}(h,v))\|^2 - \|\hat{q}(h,v)\|^2 &= \|\hat{\Lambda}(2h,v/2)\hat{q}(h,v)\|^2 - \|\hat{q}(h,v)\|^2 \\ &= \|\hat{\Lambda}(2h,v/2)\|^2\|\hat{q}(h,v)\|^2 - \|\hat{q}(h,v)\|^2 \\ &= (\|\hat{\Lambda}(2h,v/2)\|^2 - 1)\|\hat{q}(h,v)\|^2 \qquad (3.24)\end{aligned}$$

since $\|\hat{\Lambda}(2h,v/2)\|^2 = 1/(2h+v/2)^2 < 1$ as soon as $h > 0$ the expression (3.23) has a negative value which proves that the GbB-dynamical system is non-conservative.

Case 2: v is odd

Here we have

$$\begin{aligned}\|\hat{\Gamma}(\hat{q}(h,v))\|^2 - \|\hat{q}(h,v)\|^2 &= \|\hat{\Lambda}(2h+1,(v-1)/2)\hat{q}(h,v)\|^2 - \|\hat{q}(h,v)\|^2 \\ &= \|\hat{\Lambda}(2h+1,(v-1)/2)\|^2\|\hat{q}(h,v)\|^2 \\ &\qquad - \|\hat{q}(h,v)\|^2 \qquad (3.25) \\ &= (\|\hat{\Lambda}(2h+1,(v-1)/2)\|^2 - 1)\|\hat{q}(h,v)\|^2\end{aligned}$$

and $\|\hat{\Lambda}(2h+1,(v-1)/2)\|^2 = 1/((2h+1)+(v-1)/2)^2 < 1$ as soon as $h > 0$. Therefore we can make the same conclusion as in the Case 1.

The condition $h > 0$ in our proof of above limits our construction of a dissipative dynamical systems to initial states q for which $q(\cdot,y)$ is for any y not constant, which is equivalent of saying that $\hat{q}(h,v) = 0$ for $h = 0$. This requirement does however not limit the functionality of our construction.

7. Conclusion

We have shown that the Gibbs differential operator G allows the construction of a Λ - transform in the sense of Prigogine. Together with the baker transform B we can define with Λ the state transition function $\Gamma := \Lambda B$ of an autonomous discrete time dissipative dynamical system with an infinite-dimensional state space which is given the Hilbert space $L_2(E)$. For the investigation of this dynamical system Walsh harmonic analysis can successfully be applied. It was not to expect that the reached dynamical system shows any practical application in the physical sciences. However it can serve as an example to give directions for possible further research which might come closer to results which have a practical interpretation. This means that other types of non-linear state transition function, which have the mixing property, together with suitable T-operators in the sense of Misra and related Λ - transforms have to be studied to define such dynamical system which have a dissipative behaviour.

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Chapter 4

GENERALIZED INTEGRALS IN WALSH ANALYSIS

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Abstract Generalized integrals with respect to multidimensional dyadic basis are considered and applied to recover coefficients of multiple series in Haar and Walsh systems on $[0, 1]^m$ and on group G^m

1. Introduction

In this paper we survey some results related to the problem of recovering the coefficients of multiple Walsh and Haar series. Generalized integrals which solve this problem are defined in terms of the dyadic derivation basis.

It is known that similarly to the case of series in trigonometrical system (see [17]), Walsh and Haar series being convergent everywhere can fail to be the Fourier-Lebesgue series of their sums. Therefore the coefficients problem requires integration processes more general than the Lebesgue one.

A history of this theory, especially in the one-dimensional case, was presented in [14].

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We concentrate here on the multidimensional case. In this case a solution of the coefficients problem essentially depends on the type of convergence of multiple series, on the set of convergence and also on the domain on which Walsh system is defined (on the dyadic group or on the unit interval in $\mathbf{R}^m$).

A version of multidimensional Perron integral solving the coefficients problem for rectangular convergent multiple Haar and Walsh series in $\mathbf{R}^m$ was considered in [13]. We discuss this case of convergence in more details in Section 5, both in group and in $\mathbf{R}^m$ setting. The coefficient problem in the case of a more general ρ-regular rectangular convergence was considered in several paper by the first author. These results are surveyed in Section 6.

2. Henstock- and Perron-type Integrals with Respect to a Derivation Basis

We remind the principal elements of the Henstock theory of integration (see [3]).

A *derivation basis* (or simply a *basis*) $\mathcal{B}$ in a measure space $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of measurable subsets of X having positive measure μ and called *generalized intervals* or *$\mathcal{B}$-intervals*. That is, $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ so that each $\beta \in \mathcal{B}$ is a set of pairs (I, x), where $I \in \mathcal{I}$, $x \in X$, and $\mathcal{B}$ has the *filter base property*: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So each basis is a directed set with the order given by "reversed" inclusion. We shall refer to the elements β of $\mathcal{B}$ as *basis sets*. We suppose that $x \in I$ for all the pairs (I, x) constituting each $\beta \in \mathcal{B}$. For a set $E \subset X$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) := \{(I, x) \in \beta : I \subset E\} \text{ and } \beta[E] := \{(I, x) \in \beta : x \in E\}.$$

Certain additional hypotheses guarantee some nice properties of a basis. For example, it is useful to suppose that the basis $\mathcal{B}$ *ignores no point*, i.e., $\beta[\{x\}] \neq \emptyset$ for any point $x \in X$ and for any $\beta \in \mathcal{B}$.

If X is a metric or a topological space it is supposed that $\mathcal{B}$ is a *Vitali basis* by which we mean that for any x and for any neighborhood $U(x)$ of x there exists $\beta_x \in \mathcal{B}$ such that $I \subset U(x)$ for each pair $(I, x) \in \beta_x$.

A *β-partition* is a finite collection π of elements of β, where the distinct elements (I', x') and (I'', x'') in π have I' and I'' disjoint (or at least non-overlapping, i.e., $\mu(I' \cap I'') = 0$). Let $L \in \mathcal{I}$. If $\pi \subset \beta(L)$ then π is called β-partition *in L,* if $\bigcup_{(I,x)\in\pi} I = L$ then π is called β-partition *of L.*

We say that a basis $\mathcal{B}$ has the *partitioning property* if the following conditions hold: (i) for each finite collection $I_0, I_1, ..., I_n$ of $\mathcal{B}$-intervals with $I_1, ...I_n \subset I_0$ the difference $I_0 \setminus \bigcup_{i=1}^{n} I_i$ can be expressed as a finite union of pairwise non-overlapping $\mathcal{B}$-intervals; (ii) for each $\mathcal{B}$-interval I and for any $\beta \in \mathcal{B}$ there exists a β-partition of I.

DEFINITION 4.1 Let $\mathcal{B}$ be a basis having the partitioning property and $L \in \mathcal{I}$. A function f on L is said to be *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if for every $\varepsilon > 0$, there exists $\beta \in \mathcal{B}$ such that for any β-partition π of L we have:

$$\left| \sum_{(I,x)\in\pi} f(x)\mu(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

It is easy to check that if a function f is $H_{\mathcal{B}}$-integrable on L, then it is also integrable on each $\mathcal{B}$-subintervals of L and so the indefinite $H_{\mathcal{B}}$-integral is defined as an additive $\mathcal{B}$-interval function.

The following extension of the previous definition is useful in many cases.

DEFINITION 4.2 A function f defined almost everywhere on $L \in \mathcal{I}$ is *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if the function

$$f_1(x) := \begin{cases} f(x), & \text{if it is defined,} \\ 0, & \text{otherwise,} \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and its $H_{\mathcal{B}}$-integral is equal A.

Let F be an additive set function on $\mathcal{I}$ and E an arbitrary subset of X. For a fixed $\beta \in \mathcal{B}$, we set

$$Var(E, F, \beta) := \sup_{\pi\subset\beta[E]} \sum |F(I)|.$$

We put also

$$V_F(E) = V(E, F, \mathcal{B}) := \inf_{\beta\in\mathcal{B}} Var(E, F, \beta).$$

The extended real-valued set function $V_F(\cdot)$ is called *variational measure* generated by F, with respect to the basis $\mathcal{B}$. It is an outer measure and, in the case of a metric space X, a metric outer measure (in the last case it should be assumed that the basis is a Vitali basis).

Given a set function $F : \mathcal{I} \to \mathbf{R}$ we define the *upper* and *lower $\mathcal{B}$-derivative* at a point x, with respect to the basis $\mathcal{B}$ and measure μ, as

$$\overline{D}_{\mathcal{B}}F(x) := \inf_{\beta\in\mathcal{B}} \sup_{(I,x)\in\beta} \frac{F(I)}{\mu(I)} \quad \text{and} \quad \underline{D}_{\mathcal{B}}F(x) := \sup_{\beta\in\mathcal{B}} \inf_{(I,x)\in\beta} \frac{F(I)}{\mu(I)}, \quad (4.1)$$

respectively. As we have assumed that $\mathcal{B}$ ignores no point then it is always true that $\overline{D}_{\mathcal{B}}F(x) \geq \underline{D}_{\mathcal{B}}F(x)$. If $\overline{D}_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x)$ we call this common value *$\mathcal{B}$-derivative* $D_{\mathcal{B}}F(x)$.

We say that a set function F is *$\mathcal{B}$-continuous* at a point x, with respect to the basis $\mathcal{B}$, if $V_F(\{x\}) = 0$.

We shall need the following (see [3, Proposition 1.6.4])

PROPOSITION 4.1 *Let an additive function $F : \mathcal{I} \to \mathbf{R}$ be $\mathcal{B}$-differentiable on $L \in \mathcal{I}$ outside a set $E \subset L$ such that $V_F(E) = 0$. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x), & \textit{if it exists}, \\ 0, & \textit{if } x \in E, \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

The next theorem is a corollary of the above proposition.

THEOREM 4.1 *Let an additive function $F : \mathcal{I} \to \mathbf{R}$ be $\mathcal{B}$-differentiable everywhere on $L \in \mathcal{I}$ outside of a set E with $\mu(E) = 0$, and $-\infty < \underline{D}_{\mathcal{B}}F(x) < \overline{D}_{\mathcal{B}}F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}$-continuous. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x), & \textit{if it exists}, \\ 0, & \textit{if } x \in E, \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

To define a Perron-type integral with respect to a basis $\mathcal{B}$ we remind that a $\mathcal{B}$-interval function F is called *$\mathcal{B}$-superadditive* (resp. *$\mathcal{B}$-subadditive*) if every finite collection $\{I_i\}_{i=1}^p$ of pair-wise non-overlapping $\mathcal{B}$-intervals such that $\bigcup_{i=1}^p I_i \in \mathcal{I}$ satisfies

$$\sum_{i=1}^{p} F(I_i) \le F\left(\bigcup_{i=1}^{p} I_i\right) \ \text{(resp. } \sum_{i=1}^{p} F(I_i) \ge F\left(\bigcup_{i=1}^{p} I_i\right) \ \text{)}.$$

By $\overline{\mathcal{A}}_{\mathcal{B}}$ (resp. $\underline{\mathcal{A}}_{\mathcal{B}}$) denote the set of all $\mathcal{B}$-superadditive (resp. $\mathcal{B}$-subadditive) functions. A $\mathcal{B}$-interval function F is called *$\mathcal{B}$-additive* if $F \in \overline{\mathcal{A}}_{\mathcal{B}} \cap \underline{\mathcal{A}}_{\mathcal{B}}$. Let $\mathcal{A}_{\mathcal{B}}$ denote the set of all $\mathcal{B}$-additive functions.

With this notation we introduce the following definition of a Perron-type integral.

DEFINITION 4.3 A function f defined on $L \in \mathcal{I}$ is said to be *$P_{\mathcal{B}}$-integrable* on L if for every $\varepsilon > 0$ there exist $\mathcal{B}$-interval functions $M \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $m \in \underline{\mathcal{A}}_{\mathcal{B}}$ such that

$$\underline{D}_{\mathcal{B}}M(x) \ge f(x) \ge \overline{D}_{\mathcal{B}}m(x) \ \text{ for each } \ x \in L \tag{4.2}$$

and $M(L) - m(L) < \varepsilon$. The value of the *$P_{\mathcal{B}}$-integral* on L is $(P_{\mathcal{B}}) \int_L f := \inf_M M(L) = \sup_m m(L)$.

This integral is known (see [3]) to be equivalent to the $H_{\mathcal{B}}$-integral. If we want the inequality 4.2 in this definition to hold not everywhere but with some exceptional set then we need to assume some kind of continuity of the functions M and m on this set. We consider below several generalizations of the $P_{\mathcal{B}}$-integral in this direction.

3. Dyadic Derivation Bases in $[0,1]^m$ and in Group G^m

We consider here derivation bases in two spaces, according to two types of domains on which the Walsh system can be defined. The first one is the unit cube $[0,1]^m$ in which we consider the dyadic basis. Another example will be a basis in the dyadic group G or in its cartesian product G^m.

In the case of $X = [0,1]$ the family $\mathcal{I}$ of $\mathcal{B}$-intervals is constituted by dyadic intervals

$$J_j^{(n)} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \ \ 0 \le j \le 2^n - 1, \ \ n = 0, 1, 2, \dots .$$

Here n is a *rank* of the interval.

If $X = [0,1]^m$, $\mathcal{B}$-intervals are defined as m-dimensional dyadic intervals

$$J_{\mathbf{j}}^{(\mathbf{n})} := J_{j_1}^{(n_1)} \times \ldots \times J_{j_m}^{(n_m)} \tag{4.3}$$

where $\mathbf{j} = (j_1, \dots, j_m)$ and $\mathbf{n} = (n_1, \dots, n_m)$, with $\mathbf{n}$ being a rank of the interval. We denote the family of all these intervals by $\mathcal{I}_d$.

To define a dyadic basis it is enough to define basis sets β. For $X = [0,1]$ we put

$$\beta_\delta := \{I \in \mathcal{I}_d : \ \ I \subset U(x, \delta(x))\},$$

where δ is a so-called gauge, i.e., a positive function defined on X, and $U(x,\delta)$ denotes the neighborhood of x of radius δ. So the *dyadic basis* is defined as $\mathcal{B}_d := \{\beta_\delta : \ \ \delta : X \to (0,\infty)\}$.

In the m-dimensional case we consider two dyadic basis. The first one is defined exactly as above with $\mathcal{I}_d$ being the family of all m-dimensional dyadic intervals. The second one is called a regular dyadic basis. To define it we use the notion of regularity. The *parameter of regularity of a dyadic interval* of the form (4.3) is defined as

$$\min_{i,l}\{|J_{j_i}^{(n_i)}|/|J_{j_l}^{(n_l)}|\}.$$

Analogously the *parameter of regularity of a vector* $\mathbf{a} = (a_1, \dots, a_m)$ is defined as

$$\min_{i,l}\{a_i/a_l\}.$$

We write $reg(J)$ (resp. $reg(\mathbf{a})$) for the parameter of regularity of a dyadic interval J (resp. of a vector $\mathbf{a}$).

Now basis sets of *ρ-regular dyadic basis $\mathcal{B}_{d,\rho}$* we define as

$$\beta_{\delta,\rho} := \{I \in \mathcal{I}_d : \ \ I \subset U(x, \delta(x)), \ \ reg(I) \geq \rho\}\,.$$

Applying Definition 4.1 to these dyadic bases we obtain $H_{\mathcal{B}_d}$*-integral* (the *dyadic Henstock integral*) and $H_{\mathcal{B}_{d,\rho}}$*-integral* (the *ρ-regular dyadic Henstock integral*).

Now we turn to a group setting.

Recall (see [1, 2, 11]) that the *dyadic group* G is a set of sequences $t = \{t_i\}_{i=0}^{\infty}$ where $t_i = 0$ or 1 with group operation in G being defined as the coordinate-wise addition $(mod\, 2)$. The topology in G is defined by a chain of subgroups $G_k = \{t = \{t_i\} : t_i = 0, i \leq k\}, \ \ k = 0, 1, \ldots$, so that $G = G_0$ and $\{0\} = \bigcap_{n=0}^{\infty} G_n$. With respect to this topology, the subgroups G_n are clopen sets and G is a zero-dimensional compact abelian group. The factor group G/G_n contains 2^n elements. We denote by K_n any coset of the subgroup G_n and by $K_n(a)$ the coset of the subgroup G_n which contains an element $a = \{a_i\}_{i=0}^{\infty}$, i.e., $K_n(a) := a + G_n = \{t = \{t_i\} : t_i = a_i, i \leq k\}$. In particular $G_n = K_n(0)$. For each $a \in G$ the sequence $\{K_n(a)\}$ is decreasing and $\{a\} = \bigcap_n K_n(a)$.

In the product space G^m we consider, similarly to the case of the m-dimensional cube, two types of $\mathcal{B}$-intervals. By $\mathcal{I}_{G^m}$ we denote a family of all the sets of the form

$$K_{\mathbf{n}} := K_{n_1} \times \ldots \times K_{n_m}$$

where $\mathbf{n} = (\mathbf{n_1}, \ldots, \mathbf{n_m})$ is a *rank* of this $\mathcal{B}$-interval. If $\mathbf{t} \in G^m$ then

$$K_{\mathbf{n}}(\mathbf{t}) := K_{n_1}(t_1) \times \ldots \times K_{n_m}(t_m)$$

If we assume here that $reg(\mathbf{n}) \geq \rho$ for some $\rho \in (0, 1]$, then we obtain the family $\mathcal{I}^{\rho}_{G^m}$ of ρ-regular $\mathcal{B}$-intervals. Accordingly we get two derivation bases in G^m. A basis $\mathcal{B}_{G^m}$ is constituted by basis sets

$$\beta_{\nu} := \{(I, \mathbf{t}) : \mathbf{t} \in G^m, \ \ I = K_{\mathbf{n}}(\mathbf{t}), \ \ \min n_i \geq \nu(\mathbf{t})\}$$

where ν runs over the set of all integer-valued functions $\nu : G^m \to \mathbf{N}$ and $\mathbf{n} = (n_1, \ldots, n_m)$. A ρ-regular basis $\mathcal{B}^{\rho}_{G^m}$ is constituted by basis sets

$$\beta^{\rho}_{\nu} := \{(I, \mathbf{t}) \in \beta_{\nu} : I \in \mathcal{I}^{\rho}_{G^m}\}.$$

These two bases have all the properties of a general derivation basis. The partitioning property follows easily from compactness of any $\mathcal{B}_{G^m}$-interval by standard methods (see [3]).

Using the normalized Haar measures on the group G we denote by μ the product measure on G^m. Then $\mu(K_{\mathbf{n}}) = 2^{-(n_1+\ldots+n_m)}$ where $\mathbf{n} = (n_1, \ldots, n_m)$.

Definition 4.1 of the $H_{\mathcal{B}}$-integral can be rewritten for the bases in G^m in the following form (see [15]):

DEFINITION 4.4 Let $L \in \mathcal{I}_{G^m}$. A function f defined on L is said to be H_{G^m}*-integrable* (resp. $H^{\rho}_{G^m}$*-integrable*) on L, with integral value A, if for every $\varepsilon > 0$, there exists a function $\nu : L \to \mathbf{N}$ such that for any β_ν-partition (resp. β^{ρ}_{ν}-partition) π of L we have:

$$\left| \sum_{(I,\mathbf{t})\in\pi} f(\mathbf{t})\mu(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{G^m}) \int_L f$
(resp. by $(H^{\rho}_{G^m}) \int_L f$).

Note that in the case of our basis $\mathcal{B}_G$, given a point $\mathbf{t}$, any β_ν-partition contains only one pair $(I, \mathbf{t})$ with this point $\mathbf{t}$. Because of this we can reformulate the definition of $\mathcal{B}$-continuity in a simpler way, saying that a set function F is $\mathcal{B}_G$*-continuous* at a point $\mathbf{t}$, with respect to the basis $\mathcal{B}_G$, if $\lim_{n\to\infty} F(K_n(\mathbf{t})) = \mathbf{0}$.

The map

$$\Phi : t \mapsto x = \sum_{j=1}^{\infty} \frac{t_i}{2^{i+1}} \tag{4.4}$$

is one-to-one correspondence between the group G and the interval $[0,1]$, up to a countable set. Indeed, denoting by Q_d the set of all dyadic rational points in $[0,1]$, i.e., points of the form $\frac{j}{2^k}$, $0 \le j \le 2^k$, $k = 0, 1, \ldots$, we note that each $x \in Q_d$ has two expansions, a finite one and an infinite one. If we exclude from G the elements corresponding to one type of expansion, for example to the infinite one, then the correspondence (4.4) is one-to-one and the converse mapping Φ^{-1} is defined on [0,1). The function Φ maps each $\mathcal{B}_{G^m}$-interval K_n onto a dyadic interval $J^{(n)}_j$. So there is a closed relation between bases $\mathcal{B}_d$ and $\mathcal{B}_{G^m}$. But as we shall see below, the fact that G^m is a zero-dimensional space while $[0,1]^m$ is connected, implies an essential difference in the properties of the integrals defined with respect to those bases. The principal difference can be seen already in the one-dimensional case. In the case of G, we can associate with each point $t \in G$, a unique sequence of nested $\mathcal{B}_G$-intervals $K_n(t)$ converging to t. We call it the *basic sequence convergent to* t. But in the case of $X = [0,1]$ such a unique sequence of $\mathcal{B}_d$-intervals can be associated with a point x only in the case x is dyadic-irrational. If $x \in Q_d$, then we can associate with it two basic sequences of dyadic intervals: the left one and the right one for which x is the common end-point, starting with some rank n.

In the product space G^m or $[0,1]^m$ the m-multiple sequence $\{I_{\mathbf{n}}\}$ of $\mathcal{B}$-intervals is a *basic sequence convergent to* $\mathbf{t} \in G^m$ (resp. *to* $\mathbf{x} \in [0,1]^m$) if $I_{\mathbf{n}} = I_{n_1} \times \ldots \times I_{n_m}$ with $\{I_{n_i}\}$ being the one-dimensional basic sequence convergent to $t_i \in G$ (resp. to $x_i \in [0,1]$). Accordingly in G^m we have only one basic sequence convergent to each $\mathbf{t}$ while in $[0,1]^m$ the number of basic sequences convergent to $\mathbf{x}$ is equal 2^s, $0 \le s \le m$, if $\mathbf{x}$ has s dyadic-rational coordinates.

4. Multiple Walsh and Haar Series

The *Walsh functions* (in Paley numeration) on G (see [2, 11]) are defined by

$$w_n(t) := (-1)^{\sum\limits_{i=0}^{\infty} t_i \varepsilon_i^{(n)}}$$

where

$$t = \{t_i\} \in G, \qquad n = \sum_{i=0}^{\infty} 2^i \varepsilon_i^{(n)} \quad (\varepsilon_i^{(n)} \in \{0,1\}).$$

Using mapping Φ^{-1} considered above, we can define Walsh system on the unit interval as $w(\Phi^{-1}(x))$. For these functions we shall use the same notation: $w(x)$.

The *Haar functions* are usually considered on $[0,1)$. But in case of need we can always pass to the group setting using the same mapping Φ. We put $\chi_0(x) \equiv 1$. If $n = 2^k + i$ ($k = 0, 1, \ldots$, $i = 0, \ldots, 2^k - 1$), we put

$$\chi_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in \left[\frac{2i}{2^{k+1}}, \frac{2i+1}{2^{k+1}}\right), \\ -2^{k/2}, & \text{if } x \in \left[\frac{2i+1}{2^{k+1}}, \frac{2i+2}{2^{k+1}}\right), \\ 0, & \text{if } x \in [0,1) \setminus \left[\frac{2i}{2^{k+1}}, \frac{2i+2}{2^{k+1}}\right). \end{cases}$$

An *m-dimensional Walsh* (resp. *Haar*) *series* (both on G^m and on $[0,1]^m$) is defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} b_{n_1,\ldots,n_m} \prod_{i=1}^{m} w_{n_i}(x_i) \tag{4.5}$$

$$(\text{resp.} \quad \sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} a_{n_1,\ldots,n_m} \prod_{i=1}^{m} \chi_{n_i}(x_i) \ ) \tag{4.6}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real numbers. If $\mathbf{N} = (N_1, \ldots, N_m)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of series (4.5) (resp. (4.6)) at a point $\mathbf{x} = (x_1, \ldots, x_m)$

is

$$S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} b_{\mathbf{n}} \omega_{\mathbf{n}}(\mathbf{x}) \quad (\text{resp. } S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} a_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) \ ).$$

The series (4.5) (or (4.6)) *rectangularly converges* to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \text{ as } \min_i \{N_i\} \to \infty.$$

We consider also the regular convergence of series. Let $\rho \in (0, 1]$; then the series (4.5) (or (4.6)) *ρ-regularly converges* to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \text{ as } \min_i \{N_i\} \to \infty \text{ and } reg(\mathbf{N}) \geq \rho.$$

It is obvious that if the series (4.5) (or (4.6)) rectangularly converges to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ then for every $\rho \in (0, 1]$ this series ρ-regularly converges to $S(\mathbf{x})$ at $\mathbf{x}$.

A starting point for an application of the dyadic derivative and the dyadic integral to the theory of Walsh and Haar series is an observation that due to martingale properties of the partial sums $S_{2^{\mathbf{k}}}$ of those series (here $2^{\mathbf{k}}$ stand for $(2^{k_1}, \dots, 2^{k_m})$) the integral $\int_{I^{(\mathbf{k})}} S_{2^{\mathbf{k}}}$ where $I^{(\mathbf{k})}$ is a $\mathcal{B}$-interval of rank k either in $[0, 1]^m$ or in G^m, defines an additive $\mathcal{B}$-interval function $\psi(I)$ on the family $\mathcal{I}$ of all $\mathcal{B}$ intervals. (In dyadic analysis the function ψ is sometimes referred to as *quasi-measure* (see [11, 16]).) Since the sum $S_{2^{\mathbf{k}}}$ is constant on each $I^{(\mathbf{k})}$ (in the interior of $I^{(\mathbf{k})}$ in the case of $[0, 1]^m$) we get

$$S_{2^{\mathbf{k}}}(\mathbf{x}) = \frac{1}{|I^{(\mathbf{k})}|} \int_{I^{(\mathbf{k})}} S_{2^{\mathbf{k}}} = \frac{\psi(I^{(\mathbf{k})})}{|I^{(\mathbf{k})}|} \tag{4.7}$$

for any point $\mathbf{x} \in I^{(\mathbf{k})}$.

Another simple observation which is essential for proving that a given Walsh or Haar series is the Fourier series in the sense of some general integral, is the following statement (see [14, Proposition 4]).

PROPOSITION 4.2 *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}_d$ or $\mathcal{I}_{G^m}$. Assume a series of the form* (4.5) *or* (4.6) *is given. Let the $\mathcal{B}$-interval function ψ be defined for this series by* (4.7). *Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_{\mathcal{I}} \{$ for any $\mathcal{B}$-interval I.*

It is seen from formula (4.7) that for any point $\mathbf{x}$ in G^m or at least for points with dyadic-irrational coordinates, in the case of $[0, 1]^m$, rectangular (respectively, ρ-regular rectangular) convergence of the series 4.5 (or (4.6)) at a point $\mathbf{x}$ to a sum $f(\mathbf{x})$ implies $\mathcal{B}$-differentiability (respectively, $\mathcal{B}_\rho$-differentiability) of the function ψ in $\mathbf{x}$ with $f(\mathbf{x})$ being the value of the $\mathcal{B}$-derivative (resp. $\mathcal{B}_\rho$-derivative).

So in order to solve the coefficient problem it is enough to show that the function ψ is an integral of its derivative which exists at least almost everywhere. Then in view of Proposition 4.2 we get

THEOREM 4.2 *If the series* (4.5) (*or* (4.6)) *is rectangular* (*respectively, ρ-regular rectangular*) *convergent to a sum f almost everywhere on* $[0,1])^m$ *or on G^m, outside a set E such that $V_\psi(E) = 0$, then the function f is $H_{\mathcal{B}}$-integrable* (*respectively, $H_{\mathcal{B}_\rho}$-integrable and* (4.5) (*or* (4.6)) *is the Fourier series of f, in the sense of the respective integral.*

To use this theorem we need some additional information related to the behavior of a series on the exceptional set which would imply that the variational measure V_ψ is equal zero on this set. Such a nice behavior of ψ on the exceptional set can be obtained either from a convergence condition or from some additional growth assumptions imposed on the series. For example, it can be easily shown, in the one-dimensional case, that if the coefficients of a series 4.5 satisfy the condition $\lim_{n\to\infty} b_n = 0$ (which is a consequence of the convergence of the series at least at one dyadic-irrational point) then ψ is $\mathcal{B}_d$-continuous everywhere on $[0,1]$, and we apply Theorem 4.1 to get

THEOREM 4.3 *If the series* (4.5) (*in one dimension*) *is convergent to a sum f at each dyadic irrational point of* $[0,1]$, *then f is $H_{\mathcal{B}_d}$-integrable and* 4.5 *is the $H_{\mathcal{B}_d}$-Fourier series of f, i.e.,*

$$b_n = (H_{\mathcal{B}_d}) \int_{[0,1]} f w_n.$$

5. Coefficients Problem for Rectangular Convergent Series

In the case of the group setting, the equality (4.7) establishes the equivalence of rectangular convergence of the series (4.5) and (4.6) with respect to subsequence $2^{\mathbf{k}}$ and $\mathcal{B}_G$-differentiability of the associated function ψ *at each point of G*. So the problem of recovering the coefficients of everywhere convergent series is reduced in this case to the problem of recovering the primitive from the $\mathcal{B}_G$-derivative. So in this case we have

THEOREM 4.4 *If the series* (4.5) (*or* (4.6)) *is rectangular convergent to a sum f everywhere on G^m, then the function f is H_G-integrable and* (4.5) (*or* (4.6)) *is the Fourier series of f, in the sense of the H_G-integral.*

If we consider the series (4.5) and (4.6) on $[0,1]^m$ then the rectangular convergence everywhere does not guarantee the differentiability of the function ψ

everywhere. This function can fail to be differentiable on the set of points having at least one dyadic-rational coordinate. This exceptional set is not countable. So we can not apply Theorem 4.1 to get a multidimensional generalization of Theorem 4.3. Moreover it can be shown that $\mathcal{B}_d$-continuity of ψ on the exceptional set which follows from the convergence of the series is not enough to solve the problem of recovering the primitive.

But in this case a stronger type of continuity can be proved, namely the continuity in the sense of Saks.

DEFINITION 4.5 A $\mathcal{B}_d$-interval function ψ is called *continuous in the sense of Saks* if $\lim \psi(I) \to 0$ as $|I| \to 0$.

The next statement follows from [12].

PROPOSITION 4.3 *Suppose the series 4.5 everywhere rectangularly converges to a finite sum. Then the function ψ constructed for this series by 4.7 is continuous in the sense of Saks.*

Unfortunately continuity in the sense of Saks at the points of an exceptional set is not enough to recover the primitive by $H_{\mathcal{B}_d}$-integral. A reason for this is a fact that continuity of a function ψ in the sense of Saks on a set of measure zero does not imply that the variational measure V_ψ of this set is equal zero. So we can not use Theorem 4.4. Nevertheless the problem can be solved by a Perron-type integral which is a generalization of $P_{\mathcal{B}_d}$-integral.

DEFINITION 4.6 A function f defined on $[0,1]^m$ is said to be $\overline{P}_{\mathcal{B}_d}$*-integrable* if for every $\varepsilon > 0$ there exist $\mathcal{B}_d$-interval functions $F_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2 \in \underline{\mathcal{A}}_{\mathcal{B}}$ such that

(A) for each $\mathbf{x}$ with all dyadic-irrational coordinates

$$\underline{D}_{\mathcal{B}} F_1(\mathbf{x}) \geq f(\mathbf{x}) \geq \overline{D}_{\mathcal{B}} F_2(\mathbf{x});$$

(B) F_1 and F_2 are continuous in the sense of Saks everywhere on $[0,1]^m$;

(C) $F_1([0,1]^m) - F_2([0,1]^m) < \varepsilon$.

$\overline{P}_{\mathcal{B}_d}$*-integral* of the function f on $[0,1]^m$ is defined as

$$(\overline{P}_{\mathcal{B}_d}) \int_I f := \inf_{F_1} F_1([0,1]^m) = \sup_{F_2} F_2([0,1]^m).$$

THEOREM 4.5 *If the series* (4.5) *or* (4.6) *is rectangular convergent to a sum f everywhere on* $[0,1]^m$*, then the function f is* $\overline{P}_{\mathcal{B}_d}$*-integrable and* (4.5) *or* (4.6) *is the Fourier series of f, in the sense of the* $\overline{P}_{\mathcal{B}_d}$*-integral.*

6. Coefficients Problem for Regular Rectangular Convergent Series

Continuity in the sense of Saks can not be used to solve the problem in the case of regular convergence. This is clear from the following result (see [10]).

PROPOSITION 4.4 *For every* $\rho \in (0,1]$ *there is a double Walsh series* ρ*-regularly convergent to a finite sum everywhere on* $[0,1]^m$*, but* $\mathcal{B}_d$*-interval function* ψ *constructed for this series by* (4.7) *is not continuous in the sense of Saks.*

We shall use here another type of continuity at the points with dyadic-rational coordinates

In [4] the problem of recovering the coefficients of everywhere convergent double Haar series was considered. In that paper a Perron-type integral was constructed. We introduce a modified version of this integral.

DEFINITION 4.7 We say that a finite function f defined on $[0,1]^2$ is $(P_d^{1/2})$-*integrable* if for every $\varepsilon > 0$ there exist $\mathcal{B}_d$-interval functions $F_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2 \in \underline{\mathcal{A}}_{\mathcal{B}}$ with the following properties:
(A) if $\mathbf{t} = (t_1,t_2) \in ([0,1]\setminus Q_d) \times ([0,1]\setminus Q_d)$ and $\{I_{k_1,k_2}\}$ is the basic sequence convergent to $\mathbf{t}$, then

$$\varliminf_{k\to\infty} \frac{F_1(I_{k,k})}{|I_{k,k}|} \geq f(\mathbf{t}) \geq \varlimsup_{k\to\infty} \frac{F_2(I_{k,k})}{|I_{k,k}|};$$

(B) if $\mathbf{t} = (t_1,t_2) \in Q_d \times ([0,1]\setminus Q_d)$ and $\{I_{k_1,k_2}\}$ is the basic sequence convergent to $\mathbf{t}$, then

$$\lim_{k\to\infty} \frac{1}{|I_{k,k}|}\left(F_i(I_{k,k}) - \frac{1}{2}F_i(I_{k-1,k})\right) = 0 \quad (i=1,2);$$

(C) if $\mathbf{t} = (t_1,t_2) \in ([0,1]^2\setminus Q_d) \times Q_d$ and $\{I_{k_1,k_2}\}$ be the basic sequence convergent to $\mathbf{t}$, then

$$\lim_{k\to\infty} \frac{1}{|I_{k,k}|}\left(F_i(I_{k,k}) - \frac{1}{2}F_i(I_{k,k-1})\right) = 0 \quad (i=1,2);$$

(D) if $\mathbf{t} = (t_1,t_2) \in Q_d \times Q_d$ and $\{I_{k_1,k_2}\}$ be the basic sequence convergent to $\mathbf{t}$, then $\lim_{k\to\infty} \frac{1}{|I_{k,k}|}\Bigg(F_i(I_{k,k}) - \frac{1}{2}F_i(I_{k,k-1}) - \frac{1}{2}F_i(I_{k-1,k}) + \frac{1}{4}F_i(I_{k-1,k-1})\Bigg) = 0 \ (i=1,2);$
(E) $F_1([0,1]^2) - F_2([0,1]^2) < \varepsilon$.

For every dyadic interval $I \subset [0,1]^2$ we define $(P_d^{1/2})$*-integral* of the function f on I as $(P_d^{1/2}) \int_I f(t_1, t_2) := \inf_{F_1} F_1(I) = \sup_{F_2} F_2(I)$.

THEOREM 4.6 (see [4, theorem 2]) *Let* $\rho \in (0, 1/2]$ *be chosen. Suppose that a double Haar series* (4.6) *ρ-regularly converges to a finite sum* $f(t_1, t_2)$ *everywhere on* $[0,1]^2$. *Then the function f is* $(P_d^{1/2})$*-integrable and* (4.6) *is its Fourier series in the sense of the* $(P_d^{1/2})$*-integral.*

The condition $\rho \in (0, 1/2]$ in the last theorem can not be replaced by the condition $\rho = 1$. It is shown in [5] that there exists a non-trivial double Haar series convergent cubically (i.e., 1-regularly) to zero everywhere on $[0,1]^2$.

One of the properties of $(P_d^{1/2})$-integral is that this integral and Lebesgue one are incomparable [6]. But these integrals are compatible, i.e., they do not contradict to each other (see [8]). In [7] the construction of $(P_d^{1/2})$-integral was modified and the family of two-dimensional integrals was constructed. This family solves the coefficients problem for double Haar series if a special subsequence of rectangular partial sums is convergent (see [7, theorem 2]).

In [9] a generalization of $(P_d^{1/2})$-integral was introduced. We present here a modified version of this generalization.

Let Σ_m be the set of m-dimensional vectors $\sigma = (\sigma_1, \ldots, \sigma_m)$ with $\sigma_i \in \{0, 1\}$ $(i = 1, \ldots, m)$. For $\mathbf{t} = (t_1, \ldots, t_m) \in [0,1]^m$ we denote by $\Sigma_{\mathbf{t},m}$ the set of m-dimensional vectors $\sigma = (\sigma_1, \ldots, \sigma_m)$ with $\sigma_i \in \{0, 1\}$ such that if $t_i \in Q_d$, then $\sigma_i = 1$. Let $\{I_{\mathbf{k}}\}$ be a basic sequence of intervals (4.3) convergent to a point $\mathbf{t} \in [0,1]^m$. Put

$$I_{k_i}^0 = I_{k_i+1}, \quad I_{k_i}^1 = I_{k_i} \setminus I_{k_i+1}.$$

If $\sigma \in \Sigma_{\mathbf{t},m}$ or $\sigma \in \Sigma_m$ then we define by $I_{\mathbf{k}}^{\sigma}$ the dyadic interval $I_{k_1}^{\sigma_1} \times \ldots \times I_{k_m}^{\sigma_m}$. By $|\sigma|$ denote the sum $|\sigma_1| + \ldots + |\sigma_m|$.

Let $\mathbf{t} \in [0,1]^m$. We say that a function τ is Σ_m*-continuous* at a point $\mathbf{t}$ if the equation

$$\lim_{k_1 = \ldots = k_m \to \infty} \sum_{\sigma \in \mathbf{\Sigma_m}} (-1)^{|\sigma|} \tau(I_{\mathbf{k}}^{\sigma}) = 0$$

holds for any basic sequence $\{I_{\mathbf{k}}\}$ convergent to the point $\mathbf{t}$.

DEFINITION 4.8 Let f be a finite function defined on $([0,1] \setminus Q_d)^m$ except possibly on a countable set L. We say that a function f is $(P_d^{1/2,*})$*-integrable* if for every $\varepsilon > 0$ there exist $\mathcal{B}$-interval functions $F_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2 \in \underline{\mathcal{A}}_{\mathcal{B}}$ with the following properties:
(A) F_1 and F_2 are Σ_m-continuous at every point $\mathbf{t} \in [0,1]^m$;
(B) if $\mathbf{t} \in ([0,1] \setminus Q_d)^m \setminus L$ and $\{I_{\mathbf{k}}\}$ be the basic sequence of the form (4.3)

convergent to $\mathbf{t}$, then

$$\underline{\lim}_{k_1=\ldots=k_m\to\infty}\frac{F_1(I_{\mathbf{k}})}{|I_{\mathbf{k}}|}\geq f(\mathbf{t})\geq\overline{\lim}_{k_1=\ldots=k_m\to\infty}\frac{F_2(I_{\mathbf{k}})}{|I_{\mathbf{k}}|};$$

(C) if a point $\mathbf{t}\in[0,1]^m\setminus L$ has exactly $i\in\{1,\ldots,m\}$ dyadic-rational coordinates and $\{I_{\mathbf{k}}\}$ is the basic sequence convergent to $\mathbf{t}$, then

$$\lim_{k_1=\ldots=k_m\to\infty}\frac{1}{|I_{\mathbf{k}}^\sigma|^{1-i/m}}\sum_{\sigma\in\mathbf{\Sigma_{t,m}}}(-1)^{|\sigma|}F_i(I_{\mathbf{k}}^\sigma)=0\quad(i=1,2);$$

(D) $F_1([0,1]^m)-F_2([0,1]^m)<\varepsilon$.

The $(P_d^{1/2,*})$*-integral* of the function f on a dyadic interval $I\subset[0,1]^m$ is defined as $\inf_{F_1}F_1(I)=\sup_{F_2}F_2(I)$.

THEOREM 4.7 (see [9, theorem 6]). Let $\rho\in(0,1/2]$ be chosen. Suppose that the series (4.6) and some countable set $L\subset[0,1]^m$ satisfy the following conditions:
(1) for any $\mathbf{t}\in[0,1]^m$

$$b_{\mathbf{n}}\chi_{\mathbf{n}}(\mathbf{t})=\bar{\bar{o}}_{\mathbf{t}}(n_1\cdot\ldots\cdot n_m),\quad\min_i\{n_i\}\to\infty,\quad\min_{i,j}\{n_i/n_j\}\geq 1/2;\tag{4.8}$$

(2) for all $\mathbf{t}\in(I_d)^m\setminus L$ the series (4.6) ρ-regularly converges to a finite sum $f(\mathbf{t})$;
(3) if a point $\mathbf{t}\in[0,1]^m\setminus L$ has exactly $i\in\{1,\ldots,m\}$ dyadic-rational coordinates then

$$S_{\mathbf{N}}(\mathbf{t})=\bar{\bar{o}}_{\mathbf{t}}((N_1\cdot\ldots\cdot N_m)^{i/m}),\quad\min_i\{N_i\}\to\infty\quad\min_{i,j}\{N_i/N_j\}\geq 1/2.$$

Then the function f is $(P_d^{1/2,*})$-integrable and 4.6 is its Fourier series in the sense of the $(P_d^{1/2,*})$-integral.

In [10] the group G^m instead of the unit cube $[0,1]^m$ was considered. In this case a more simple integral solving the coefficients problems for both Haar and Walsh series was constructed.

DEFINITION 4.9 Let for every point $\mathbf{t}\in G^m$ except possibly a countable set L an increasing sequence of natural numbers $\{k_j=k_j(\mathbf{t})\}$ be chosen. We say that a finite function f defined on $G^m\setminus L$ is $P(k_j)$*-integrable* if for every $\varepsilon>0$ there exist $\mathcal{B}$-interval functions $F_1\in\overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2\in\underline{\mathcal{A}}_{\mathcal{B}}$ with the following properties:
(A) F_1 and F_2 are Σ_m-continuous at every point $\mathbf{t}\in G^m$;

(B) if $\mathbf{t}$ is any point of $G^m \setminus L$ and $\{I_{\mathbf{k}}\}$ is the basic sequence convergent to $\mathbf{t}$ then

$$\underline{\lim}_{j\to\infty} F_1(I_{k_j,\ldots,k_j})/|I_{k_j,\ldots,k_j}| \geq f(\mathbf{t}) \geq \overline{\lim}_{j\to\infty} F_2(I_{k_j,\ldots,k_j})/|I_{k_j,\ldots,k_j}|;$$

(C) $F_1(G^m) - F_2(G^m) < \varepsilon$.
For every dyadic interval I the $P(k_j)$-*integral* of the function f on I is defined as $\inf_{F_1} F_1(I) = \sup_{F_2} F_2(I)$.

The next theorems were proven in [10].

THEOREM 4.8 *Let at every point $\mathbf{t} \in G^m$, except possibly a countable set L, the increasing sequence of natural numbers $\{k_j = k_j(\mathbf{t})\}$ be chosen. Assume that for the series (S) of the form (4.6) the following conditions hold:*
(1) *at every point $\mathbf{t} \in \mathbf{G^m} \setminus \mathbf{L}$ the subsequence $S_{2^{k_j(\mathbf{t})},\ldots,2^{k_j(\mathbf{t})}}(\mathbf{t})$ of the cubical partial sums of the series (S) converges to a finite sum $f(\mathbf{t})$ as $j \to \infty$;*
(2) *at every point $\mathbf{t} \in G^m$ the series (S) satisfies the condition (4.8).*
Then the function $f(\mathbf{t})$ is $(P(k_j))$-integrable and the series (S) is its Fourier series in the sense of the $(P(k_j))$-integral.

A similar result holds for the Walsh series.

THEOREM 4.9 *Let at every point $\mathbf{t} \in G^m$ except possibly a countable set L an increasing sequence of natural numbers $\{k_j = k_j(\mathbf{t})\}$ be chosen. Assume that for the series (S) of the form (4.5) the following conditions hold:*
(1) *at every point $\mathbf{t} \in G^m \setminus L$ the subsequence $S_{2^{k_j(\mathbf{t})},\ldots,2^{k_j(\mathbf{t})}}(\mathbf{t})$ of the cubical partial sums of the series (S) converges to a finite sum $f(\mathbf{t})$ as $j \to \infty$;*
(2) *the series (S) satisfies the condition*

$$a_{\mathbf{n}} = \overline{\overline{o}}(1), \quad \min_i\{n_i\} \to \infty, \quad \min_{i,j}\{n_i/n_j\} \geq 1/2.$$

Then the function $f(\mathbf{t})$ is $(P(k_j))$-integrable and the series (S) is $(P(k_j))$-Fourier series of the function $f(\mathbf{t})$.

As a corollary we get

THEOREM 4.10 *Let a number $\rho \in (0, 1/2]$ be chosen. Suppose that the m-multiple Walsh or Haar series ρ-regularly converges to a finite sum $f(\mathbf{t})$ at every point $\mathbf{t} \in G^m$ except possibly a countable set L. Then for every choice of a sequence $\{k_j = k_j(\mathbf{t})\}$ the function $f(\mathbf{t})$ is $(P(k_j))$-integrable and the given series is $(P(k_j))$-Fourier series of the function $f(\mathbf{t})$.*

For more details see [7], [8], [10] and [14].

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Chapter 5

DYADIC DISTRIBUTIONS

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Abstract We introduce the space $D'_d(R_+)$ of dyadic distributions and the space $S'_d(R_+)$ of dyadic tempered distributions based on the dyadic derivative. [1]

1. Introduction

Distribution theory goes back to Sobolev [1], [2] and Schwartz [3]-[5], although the first generalized functions - for instance, the famous *Dirac delta-function* - appeared already in Dirac's papers on quantum mechanics. At present, distribution theory is well developed and is the subject of many monographs (see, for example, Gelfand and Shilov [6], Vekua [7], Antosik, Mikusinski and Sikorski [8], Zemanian [9], and other authors). Distributions have found a wide range of applications in mathematical physics, physics, quantum mechanics, and other branches of natural sciences (see, for example, Vladimirov [10], or Schwartz [3]). Distribution theory is used in the proof of some results of harmonic analysis (see, for instance, Stein [11], Ch. 3, Garcia-Cuerva and Rubio de Francia [12], Ch. 3, or Edwards [13], Ch. 3), the theory of functions of the complex variable [7], and other areas of mathematics.

In the monograph of Taibleson [14] a sketch of distribution theory on local fields is given. In the monograph of Vladimirov, Volovich and Zelenov [15] the theory of distributions on the field of p-adic numbers is developed.

Vilenkin [16] introduced the notion of the distribution on locally compact commutative group. But he restricted himself only by three concrete examples of such distributions. In particular, he did not introduce the operation of differentiation of these distributions.

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We introduce the space $D'_d(R_+)$ of *dyadic distributions* and the space $S'_d(R_+)$ of *dyadic tempered distributions* on the base of *dyadic derivative* which was introduced by Onneweer [17]. The completeness of the spaces $D'_d(R_+)$ and $S'_d(R_+)$ is proven.

2. Lemmas

Let N (or Z_+) be the set of all positive (respectively non negative) integers. For $x \in R_+$ and $n \in N$ we set

$$x_n \equiv [2^n x] \pmod{2}, \quad x_{-n} \equiv [2^{1-n} x] \pmod{2}, \tag{5.1}$$

where x_n and x_{-n} are equal to 0 or 1.

Let us introduce the distance $\rho^*(x,y)$ on R_+ as follows

$$\rho^*(x,y) = \sum_{i=1}^{+\infty} 2^{i-1}|x_{-i} - y_{-i}| + \sum_{i=1}^{+\infty} \frac{|x_i - y_i|}{2^i}, \quad x, y \in R_+.$$

It is not difficult to prove the inequalities

$$\rho(x,y) \equiv |y - x| \le \rho^*(x,y), \quad x, y \in R_+.$$

The function $f : R_+ \to R$ is called uniformly ρ^*-continuous on R_+ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \rho^*(x,y) < \delta, \quad x, y \in R_+ \Rightarrow |f(y) - f(x)| < \varepsilon.$$

Let us consider the dyadic convolution $f * g$ of two functions $f \in L(R_+)$ and $g \in L^\infty(R_+)$:

$$(f * g)(x) = \int_{R_+} f(\rho^*(x,y))g(y)dy.$$

LEMMA 5.1 *If the function f is uniformly ρ^*-continuous and bounded on R_+, then for all $g \in R_+$, the dyadic convolution $(f * g)(x)$ is also uniformly ρ^*-continuous and bounded on R_+.*

We define the function $h : (0, +\infty) \to (0, +\infty)$ by the equalities

$$h(x) = 2^n, \quad 2^n \le x < 2^{n+1}, \quad n \in .$$

Let us set

$$\Lambda_n^\alpha(x) = \int_0^{2^n} (h(t))^\alpha \psi(x,t)dt, \quad x \in R_+, \quad \alpha > 0.$$

LEMMA 5.2 *For $\alpha > 0$, $n \in Z_+$, the function Λ_n^α is bounded on R_+ and $\Lambda_n^\alpha \in L(R_+)$.*

DEFINITION 5.1 *Let $\alpha > 0$ and $f \in L(R_+) \cup L^\infty(R_+)$. If there exists the finite limit*

$$f^{(\alpha)}(x) = \lim_{\alpha \to \infty} (f * \Lambda_n^\alpha)(x),$$

at the point $x \in R_+$, then the number $f^{(\alpha)}(x)$ is called the dyadic derivative *(DD) of order α of the function f at the point x.*

This definition has been introduced by C.W. Onneweer [17].
For $(x, y) \in R_+ \times R_+$, we set

$$t(x, y) = \sum_{n=1}^{\infty} (x_n y_{-n} + x_{-n} y_n).$$

This sum is finite, because $x_{-i} = 0$ for $i \geq i(x) \in N$, (see (5.1)).
The generalized Walsh functions ψ_y $(y \in R_+)$ are defined by the equalities

$$\psi_y(x) \equiv \psi(x, y) = (-1)^{t(x,y)}.$$

Let us consider the function system

$$\varphi_{m,n}(x) \equiv \psi(x, m2^{-n})\mathbf{X}_{[0,2^n)}(x), \quad (m \in N, n \in Z). \tag{5.2}$$

LEMMA 5.3 *Each function of the system (5.2) is uniformly ρ^*-continuous on R_+.*

LEMMA 5.4 *For each $\alpha > 0$ and $x \in R_+$ there exists the DD $\varphi_{m,n}^{(\alpha)}(x)$ and*

$$\varphi_{m,n}^{(\alpha)}(x) = 2^{r\alpha} \varphi_{m,n}(x)$$

where the integer r is uniquely defined by the imbedding

$$[m2^{-n}, (m+1)2^{-n}) \subset [2^r, 2^{r+1}),$$

moreover

$$(\varphi_{m,n} * \Lambda_k^\alpha) = 2^{r\alpha} \varphi_{m,n} \quad for \quad k \geq max\{r, \log_2[(m+1)2^{-n}]\}.$$

3. The Space of Dyadic Distributions

DEFINITION 5.2 *The function φ is called infinitely dyadic smooth on R_+ if there exists the dyadic derivative $\varphi^{(\alpha)}(x)$ for all $\alpha \in N$, which is uniformly ρ^*-continuous and bounded on R_+.*

We denote by $C_W^{(\infty)}(R_+)$ the set of all such functions and by $supp\varphi$ the support of the function $\varphi : R_+ \to R$. The set

$$\Delta = [m2^{-n}, (m+1)2^{-n}), \quad m \in Z_+, \quad n \in Z,$$

is called *the dyadic interval*.

DEFINITION 5.3 *The function is called dyadic compactly supported, if:*

1. *There exists a dyadic interval Δ such that $supp\varphi^{(\alpha)} \subset \Delta \forall \alpha \in Z_+$,*

2. *For each $\alpha \in Z_+$ the sequence $(\varphi * \Lambda_n^{(\alpha)})(x)$ converges to $\varphi^{(\alpha)}(x)$ uniformly on R_+.*

The set of all dyadic compactly supported functions from $C_w^{(\infty)}(R_+)$ will be denoted by $D_d(R_+)$. By the Lemmas 5.3 and 5.4 we have $D_d(R_+) \supset \varphi_{m,n}$ (see (5.2)).

REMARK 5.1 *There exists a compactly supported function $\varphi \in C_w^{(\infty)}(R_+)$ such that $\varphi \notin D_d(R_+)$. For example, $\varphi_{0,n}(x) = X_{[0,2^n)}(x)$, $(n \in Z)$ is such a function.*

DEFINITION 5.4 *The sequence $\{\varphi_n\}_{n=1}^{+\infty} \subset D_d(R_+)$ is called convergent in the space $D_d(R_+)$ to the function $\varphi_0 \in D_d(R_+)$ if:*

1. *There exists a dyadic interval Δ such that $supp\varphi_n^{(\alpha)} \subset \Delta$ for all $n, \alpha \in Z_+$;*

2. *$\varphi_n^{(\alpha)}(x)$ converges to $\varphi_0^{(\alpha)}(x)$ uniformly on Δ for each $\alpha \in Z_+$.*

In this case we will write: $\varphi_n \to \varphi_0$ in $D_d(R_+)$.

Let us consider the functional $f : D_d(R_+) \to R$. Its value on the element $\varphi \in D_d(R_+)$ will be denoted by (f, φ).

DEFINITION 5.5 *The linear continuous functional $f : D_d(R_+) \to R$ is called* dyadic distribution.

We denote by $D'_d(R_+)$ the set of all dyadic distributions.

For any functions $f_1, f_2 \in D'_d(R_+)$ we set

$$(c_1 f_1 + c_2 f_2, \varphi) = c_1(f_1, \varphi) + c_2(f_2, \varphi), \quad \varphi \in D_d(R_+), \quad \forall c_1, c_2 \in R.$$

Then $D'_d(R_+)$ will be a linear space.

THEOREM 5.1 *Each function $f \in L_{loc}(R_+)$ generates a dyadic distribution, if we set*

$$(f, \varphi) = \int_{R_+} f(x)\varphi(x)dx, \quad \forall \varphi \in D_d(R_+). \tag{5.3}$$

Dyadic distributions of the form (5.3) are called *regular*. The remaining dyadic distributions are called *singular*.

An example of singular dyadic distribution gives *the dyadic δ- function*:

$$(\delta_d, \varphi) = \varphi(0), \quad \varphi \in D_d(R_+).$$

DEFINITION 5.6 *For $f \in D'_d(R_+)$ and $\alpha \in N$, we set*

$$(f^{(\alpha)}, \varphi) = (f, \varphi^{(\alpha)}), \quad \varphi \in D_d(R_+).$$

DEFINITION 5.7 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset D'_d(R_+)$ is called convergent to the dyadic distribution f_0, if*

$$\lim_{n \to +\infty} (f_n, \varphi) = (f_0, \varphi), \quad \forall \varphi \in D_d(R_+).$$

In this case we will write: $f_n \to f_0$ in $D'_d(R_+)$.

DEFINITION 5.8 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset L_{loc}(R_+)$ is called to be convergent to the function $f_0 \in L_{loc}(R_+)$ in the space $L_{loc}(R_+)$, if this sequence converges to the function f_0 in the space $L(\Delta)$ for all dyadic intervals Δ.*

In this case we will write $f_n \to f_0$ in $L_{loc}(R_+)$.

THEOREM 5.2 *If $f_n \to f_0$ in the space $L_{loc}(R_+)$, then $f_n \to f_0$ in $D'_d(R_+)$.*

DEFINITION 5.9 *The sequence $\{f_n\}_{n=1}^{+\infty} \subset D'_d(R_+)$ is called fundamental in $D'_d(R_+)$ if for each function $\varphi \in D_d(R_+)$ the sequence $\{(f_n, \varphi)\}_{n=1}^{+\infty}$ is fundamental in R.*

The following theorem states the completeness of the space $D'_d(R_+)$.

THEOREM 5.3 *If the sequence $\{f_n\}_{n=1}^{+\infty} \subset D'_d(R_+)$ is fundamental in $D'_d(R_+)$, then the functional defined by the equality*

$$(f, \varphi) = \lim_{n \to +\infty} (f_n, \varphi), \quad \forall \varphi \in D_d(R_+),$$

is linear and continuous on $D_d(R_+)$, e.g., $f \in D'_d(R_+)$.

4. The Space of Dyadic Tempered Distributions

Let us define the space of fast decreasing functions in the neighborhood of $+\infty$.

DEFINITION 5.10 *The function $\varphi \in C_w^{(+\infty)}(R_+)$ is said to belong to the set $S_d(R_+)$, if*

$$\lim_{x\to+\infty} x^\beta \varphi^{(\alpha)}(x) = 0, \quad \forall \alpha, \beta \in Z_+.$$

It is evident that the set $S_d(R_+)$ is a real linear space.

DEFINITION 5.11 *The sequence $\{\varphi_n\}_{n=1}^{+\infty} \subset S_d(R_+)$ is said to be convergent in the space $S_d(R_+)$ to the function $\varphi_0 \in S_d(R_+)$ if*

$$x^\beta \varphi_n^{(\alpha)}(x) \to x^\beta \varphi_0^{(\alpha)}(x), \quad n \to +\infty \quad \textit{uniformly on} \quad R_+ \quad \forall \alpha, \beta \in Z_+.$$

This fact will be written as follows: $\varphi_n \to \varphi_0$ in $S_d(R_+)$.

Let us consider the linear functional $f : S_d(R_+) \to R$. Its value on the function $\varphi \in S_d(R_+)$ will be denoted by (f, φ). The linear functional $f : S_d(R_+) \to R$ is called *continuous* if the condition "$\varphi_n \to \varphi_0$ in $S_d(R_+)$" implies

$$(f, \varphi_n) \to (f, \varphi), \quad \forall \varphi \in S_d(R_+).$$

DEFINITION 5.12 *The linear continuous functional $f : S_d(R_+) \to R$ is called the dyadic tempered distribution.*

The set of all dyadic tempered distributions will be denoted by $S'_d(R_+)$. Let us introduce the operations of addition of two dyadic tempered distributions $f_1, f_2 \in S'_d(R_+)$ and the multiplication of such distributions on real numbers as follows:

$$(c_1 f_1 + c_2 f_2, \varphi) = c_1(f_1, \varphi) + c_2(f_2, \varphi),$$

where $c_1, c_2 \in R_+$, and $\varphi \in S_d(R_+)$.

Then, the set $S'_d(R_+)$ is the real linear space which is conjugate to the space $S_d(R_+)$.

Let us recall the known terminology. The function $f : R_+ \to R$ has the polynomial growth in the neighbourhood of $+\infty$ if there exists a number $\beta \in Z$ such that $f(x) = O(x^\beta)$ as $x \to +\infty$.

THEOREM 5.4 *If a local integrable function $f(x)$ has the polynomial growth in the neighborhood of $+\infty$, then it generates the dyadic tempered distribution by the equality*

$$(f, \varphi) = \int_{R_+} f(x)\varphi(x)dx, \quad \varphi \in S_d(R_+).$$

Let us introduce the set of convergent sequences in the space $S'_d(R_+)$.

DEFINITION 5.13 *The sequence* $\{f_n\}_{n=1}^{+\infty} \subset S'_d(R_+)$ *is said to be convergent to the element* $f_0 \in S'_d(R_+)$, *if*

$$\lim_{n\to+\infty}(f_n,\varphi) = (f_0,\varphi),$$

for each function $\varphi \in S_d(R_+)$.

DEFINITION 5.14 *The sequence* $\{f_n\}_{n=1}^{+\infty} \subset S'_d(R_+)$ *is said to be fundamental in the space* $S'_d(R_+)$, *if the sequence* $\{(f_n,\varphi)\}_{n=1}^{+\infty}$ *is fundamental in* R *for each function* $\varphi \in S_d(R_+)$.

THEOREM 5.5 *The space* $S'_d(R_+)$ *is complete, i.e., if the sequence* $\{f_n\}_{n=1}^{+\infty} \subset S'_d(R_+)$ *is fundamental in the space* $S'_d(R_+)$, *then the functional* f *defined by the equality*

$$(f,\varphi) = \lim_{n\to+\infty}(f_n,\varphi), \quad \forall\varphi \in S_d(R_+),$$

belongs to the space $S'_d(R_+)$.

Let us define the operation of dyadic differentiation of a dyadic tempered distribution.

DEFINITION 5.15 *If* $f \in S'_d(R_+)$ *and* $\alpha \in N$, *then we set*

$$(f^{(\alpha)},\varphi) = (f,\varphi^{(\alpha)}), \quad \varphi \in S_d(R_+).$$

It is evident that for every $\varphi \in S_d(R_+)$ we have $\varphi^{(\alpha)} \in S_d(R_+)$, $\forall\alpha \in N$. Therefore it follows from the Definition 5.15 that

$$\forall f \in S'_d(R_+) \quad \exists f^{(\alpha)} \in S'_d(R_+) \quad \forall\alpha \in N.$$

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Chapter 6

ON DYADIC FRACTIONAL DERIVATIVES AND INTEGRALS

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Abstract The paper presents two dyadic analogs of the Lebesgue theorems on the differentiation of indefinite integral and on the integration of the derivative of a function. It is considered also the dyadic fractional integration by parts and the problem of dyadic fractional differentiation and integration of the integral depending on a real parameter. Most of these results are new also when considered dyadic derivatives and integrals of the first order. [1]

1. Introduction

The notions of pointwise and strong dyadic derivatives and integrals are known. The strong dyadic derivatives and integrals are defined on the segment $[0, 1]$ or on the positive half-line R_+ for some classes of functions. Sometimes they are defined on dyadic groups G or K, where G is isomorphic to the modified dyadic segment $[0, 1]^*$ and K is isomorphic to the modified half-line R_+^*.

P.L. Butzer and H.J. Wagner [1] introduced *the strong dyadic derivative* $D(f) \in L(G)$ for functions $f \in L(G)$. They proved that for the functions of Walsh-Paley system $\{w_n\}_{n=0}^{\infty}$ the equalities

$$D(w_n) = nw_n, \quad (n \in Z_+),$$

hold. This means that the Walsh-Paley functions are eigenfunctions of the operator D.

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In the same paper [1] for functions $f \in L(G)$ *the strong dyadic integral* $I(f) \in L(G)$ is introduced and proven are the equalities

$$D(T(f)) = I(D(f)) = f, \quad \text{if} \quad \hat{f}(0) \equiv \int_G f(x)d\mu(x) = 0,$$

where μ is the normalized Haar measure on G, i.e., $\mu(G) = 1$.

Thus, *the fundamental theorem of dyadic integral calculus* has been established. In the same paper the equivalence of the following three conditions was proven:

1 There exists $D(f) = g$;

2 There exists a function $g \in L(G)$ such that $\hat{g}(n) = n\hat{f}(n)$, $n \in Z$, where

$$\hat{f}(n) = \int_G f(x)w_n(x)d\mu(x),$$

are *Walsh - Fourier coefficients* of the function f;

3 There exists a function $g \in L(G)$ such that $f = I(G) + \hat{f}(0)$.

In another paper [2] by the same authors, the strong dyadic derivative $D(f)$ for functions $f \in L(R_+)$ is introduced and the equality $(D(f))^{\sim}(x) = x\tilde{f}(x)$ is proven, where $\tilde{f}$ is the Walsh - Fourier transformation of the function f.

For the functions $f \in L(R_+)$, H.J. Wagner [3] defined the strong dyadic integral $I(f)$ and proved the equalities

$$(D(I(f)) = f, \quad I(D(f)) = f.$$

(The last equality is proved under the condition $\tilde{f}(0) = 0$). In the same paper [3] the following criterion is proven: For a pair of functions $f, g \in L(R_+)$ the equality holds if and only if $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)/x$, $x > 0$.

In the paper [4], P.L. Butzer and H.J. Wagner investigated properties of *the pointwise dyadic derivatives* for functions defined on the segment $[0, 1]$.

C.W. Onneweer [5] introduced a *modified dyadic derivative* $D^{(1)}(f)$ for functions $f \in L(G)$ and proved the equalities

$$D^{(1)}(w_k) = 2^n w_k, \quad 2^n \leq k < 2^{n+1}, \quad n \in Z_+.$$

In the papers [5] and [6], C.W. Onneweer introduced also two strong dyadic derivatives $D^{(1)}(f)$ and $D^{[1]}(f)$ for functions $f \in L(K)$. In [6], he proved that the derivatives $D^{(1)}$ and $D^{[1]}$ have the same domain that does not coincide with $L(K)$. In the paper [7], C.W. Onneweer introduced *the pointwise p-adic derivative* $d_p^{(\alpha)}(f)(x)$ and *the strong p-adic derivative* $D^{(\alpha)}(f)$ *of fractional* *order* α for functions defined on the compact group G_p, which is the countable sum of cyclic groups of order $p \geq 2$. In this case, the strong p-adic derivative $D_p^{(\alpha)}(f)$ for functions in the spaces $L^q(G_p)$ and $C(G_p)$ is defined. C.W. Onneweer obtained the formulae for the derivatives of order α of functions that are the characters of the group G_p. It follows from this formulae that the characters of the group G_p are the eigenfunctions of the operator D_p^{α}.

In the same paper [7], *the strong p-adic integral of fractional positive order* α is introduced and the equalities

$$D_p^{(\alpha)}(I_p^{(\alpha)}(f)) = f, \quad I_p^{(\alpha)}(D_p^{(\alpha)}(f)) = f,$$

under condition $\int_{G_p} f(x)d\mu(x) = 0$ are proved. Also, a criterion is obtained in order that for the pair of functions f, g the equality $g = I_p^{(\alpha)}(f)$ or $g = D_p^{(\alpha)}(f)$ to be fulfilled.

Let us note that He Zelin [8] obtained the similar results for *the fractional derivatives and integrals of Butzer-Wagner type*.

The theory of dyadic differentiation and integration is presented explicitly in the books [9] and [10]. Some aspects of this theory are considered in the book [11].

In our paper [12] the modified strong dyadic integral $J_\alpha(f)$ and derivative $D^{(\alpha)}(f)$ of fractional positive order α have been introduced. Notice that the derivative $D^{(\alpha)}(f)$ for functions $f \in L(R_+)$ actually is a modification of the derivative of C.W. Onneweer, who considered the case $f \in L(K)$.

We formulate below two dyadic analogs of the Lebesgue theorems on the differentiation of indefinite integral and on the integration of the derivative of a function. We consider also dyadic fractional integration by parts and the problem of dyadic fractional differentiation and integration of the integral depending on a real parameter. The most results are new also for dyadic derivatives and integrals of the first order.

2. Definitions and Auxiliary Results

As usually, R_+ is the positive real axis. By $L^p(E)$, $1 \leq p \leq \infty$, we denote the space of Lebesgue measurable functions f on the measurable set $E \subset R_+$ with finite norm

$$\|f\|_{L_p(E)} = \left(\int_E |f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|f\|_{L^\infty(E)} = ess \sup_{x\in E} |f(x)|.$$

For a number $x \in R_+$ and a natural n, we set

$$x_n \equiv \lceil 2^n x \rceil \pmod 2, \quad x_{-n} \equiv \lceil 2^{1-n} x \rceil \pmod 2, \tag{6.1}$$

where $\lceil a \rceil$ is the integer part of the number a, x_n and x_{-n} are equal to 0 or 1.

Let us note that x_n (x_{-n}) is the n-th dyadic digit of the integer part (the fractional part respectively) of the number $x \in R_+$. The dyadic rational number $x \in R_+$ has finite dyadic expansion, i.e., $x_n = 0$ for $n \geq n(x)$.

As $x_{-n} = 0$ for $n \geq n(x)$, then for $(x, y) \in R_+ \times R_+$, the non negative integer

$$t(x, y) = \sum_{n=1}^{\infty} (x_n y_{-n} + x_{-n} y_n),$$

is defined.

Let us introduce *the Walsh kernel*

$$\psi(x, y) = (-1)^{t(x,y)}. \tag{6.2}$$

The Walsh - Fourier transform $F[f] \equiv \tilde{f}$ of the function $f \in L(R_+)$ is introduced by the equality

$$F[f](x) \equiv \tilde{f}(x) = \int_{R_+} \psi(x, y) f(y) dy. \tag{6.3}$$

This definition can be generalized for functions in the space $L^p(R_+)$, where $1 < p \leq 2$. In this case, the Walsh - Fourier transform $F[f] \equiv \tilde{f}$ is defined as the limit of the sequence

$$\int_0^{2^n} f(y) \psi(x, y) dy, \quad n \in Z_+,$$

by the norm of the space $L^p(R_+)$.

The properties of the Walsh - Fourier transform are similar to that of classical Fourier transform.

Let us introduce the operation of dyadic addition $\oplus$ on R_+ by setting

$$x \oplus y = z, \quad \text{for} \quad x, y \in R_0,$$

where the number z has dyadic digits $z_n = x_n + y_n \bmod 2$, $n \in Z \setminus \{0\}$ and x_n, y_n are defined by the rule (6.1).

Let us note that

$$z_n = \sum_{n=1}^{\infty} 2^{n-1} z_{-n} + \sum_{n=1}^{\infty} \frac{z_n}{2^n},$$

and the case $z_n = 1$ for $n \geq n(z)$ is not excluded.

For the Walsh kernel (6.2) the equality

$$\psi(x \oplus y, t) = \psi(x, t) \psi(y, t), \tag{6.4}$$

holds, if $t, x, y \in R_+$ and $x \oplus y$ is not a dyadic rational (see [9], p. 421). Thus for fixed t and x the equality (6.4) for almost all $y \in R_+$ is valid.

Let us note that

$$\psi(x,k) = w_k(x - \lceil x \rceil), \quad (k \in Z_+, x \in R_+),$$

where $\{w_k\}_{k=0}^{\infty}$ is *the Walsh-Paley system*, which is orthonormal on the segment $[0,1)$.

The Walsh-Fourier coefficients $\hat{f}(k)$ of the function $f \in L[0,1)$ are defined by

$$\hat{f}(k) = \int_0^1 f(x) w_k(x) dx, \quad k \in Z_+. \tag{6.5}$$

Let us remind the definition of W-continuity of a function.

The function f is called *W-continuous* at the point $x \in R_+$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x \oplus y) - f(x)| < \varepsilon \quad \text{for} \quad 0 \le y < \delta,$$

(see [11], Ch. 1).

Let us note that for each function $f \in L(R_+)$ its Walsh-Fourier transform $\tilde{f}$ is W-continuous on R_+ (see [11], theorem 6.1.5).

Let us introduce *the dyadic Lebesgue point* of a function.

The point $x \in R_+$ is called the dyadic Lebesgue point of a locally integrable function f, if this function is finite at this point and

$$\lim_{n \to +\infty} 2^n \int_0^{2^{-n}} |f(x \oplus t) - f(x)| dt = 0. \tag{6.6}$$

We generalize the notion of dyadic Lebesgue point as follows.

The dyadic segment of the rank n, $(n \in Z)$ is the set $\Delta_i^{(n)} = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ where $i \in Z_+$. The point $x \in R_+$ will be called *the binary d-point* of the function f, if $f(x)$ is finite and

$$\lim_{n \to +\infty} \frac{1}{|\Delta_i^{(n)}|} \int_{\Delta_i^{(n)}} f(t) dt = f(x), \tag{6.7}$$

where $\Delta_i^{(n)}$ is dyadic segment containing the point x and $|\Delta_i^{(n)}| = 2^{-n}$.

Every dyadic Lebesgue point of a function is also its binary d-point, i.e., (6.7) follows from (6.6).

For a locally Lebesgue integrable function f we set

$$F(x) = \int_0^x f(t) dt, \quad x \in R_+.$$

It is known that $F'(x) = f(x)$ almost everywhere on R_+ [15] and each point x for which the condition $F'(x) = f(x) \neq \pm\infty$ holds is a binary point of the function f (see [11], Ch. 1).

Thus, almost all points of the positive real axis R_+ are binary d-points of every function locally integrable on R_+. It can be proven that almost all points of the positive real axis r_+ are dyadic Lebesgue points of each function locally integrable on R_+.

For the function $f \in L[0, 1)$, we set

$$S_m(f)(x) = \sum_{k=0}^{m-1} \hat{f}(k) w_k(x),$$

i.e., $S_m(f)(x)$ is *the Walsh-Fourier sum of order* m of the function f. The following theorem is known.

THEOREM 6.1 *Let the function* $f \in L[0, 1)$ *has a finite value at the point* $x \in [0, 1)$. *Then, the condition*

$$\lim_{n\to+\infty} S_{2^n}(f)(x) = f(x),$$

holds if and only if x *is the binary* d*-point of the function* f *(see [11], p. 59).*

This statement follows from the equality

$$S_{2^n}(f)(x) = \frac{1}{|\Delta_i^{(n)}|} \int_{\Delta_i^{(n)}} f(t)dt, \quad (n \in Z_+), \tag{6.8}$$

where $\Delta_i^{(n)}$ is the dyadic segment containing the point x (see [11], p. 45).

Let us formulate an analog of the Theorem 6.1 for the functions defined on R_+. For this aim we need *the generalized Walsh-Dirichlet integral* of the function f:

$$S_y(f)(x) = \int_0^y \tilde{f}(t)\psi(x, t)dt. \tag{6.9}$$

For the integral (6.9) the equality (6.8) is also valid (see [9], p. 428). Thus, the following theorem is true.

THEOREM 6.2 *Let the function* $f \in L(R_+)$ *has a finite value at the point* $x \in [0, 1)$ *and* $S_y(f)(x)$ *is its generalized Walsh-Dirichlet integral. Then, the condition*

$$\lim_{n\to+\infty} S_{2^n}(f)(x) = f(x),$$

holds if and only if x *is a binary* d*-point of the function* f.

LEMMA 6.1 *For each $n \in Z$ and $\alpha > 0$ the function*

$$W_n^{\{\alpha\}}(x) \equiv \lim_{m\to+\infty} \int_{2^{-n}}^{2^n} \psi(x,y) y^{-\alpha} dy, \tag{6.10}$$

is well defined at each point $x > 0$, $W_n^{\{\alpha\}} \in L(R_+)$ and the limit in the right-hand side of (6.10) also exists in the metric of the space $L(R_+)$.

For $\alpha = 1$, the statement of this lemma is known (see [9], p. 434).

DEFINITION 6.1 *Let $\alpha > 0$, $f, g \in L^p(R_+)$, $1 \le p \le \infty$ and*

$$\lim_{n\to+\infty} \|f * W_n^{\{\alpha\}} - g\|_{L^p(R_+)} = 0.$$

Then, the function $g \equiv I_{\{\alpha\}}(f)$ is called the Wagner strong dyadic integral *(WSDI) of order α of the function f in the space $L^p(R_+)$.*

For $\alpha = 1$, $p = 1$, this definition was introduced by H.J. Wagner (see also [9], Ch. 9). Using this definition and 6.1 one can prove the following

THEOREM 6.3 *Let $\alpha > 0$, $f, g \in L(R_+)$. Then, the function g is the WSDI of order α of the function f in the space $L(R_+)$ if and only if*

$$\tilde{g}(0) = 0 \quad \textit{and} \quad \tilde{g}(x) = \tilde{f}(x) x^{-\alpha} \quad \textit{for} \quad x > 0.$$

For $\alpha = 1$ this theorem was proven by H.J. Wagner [3] (see also [9], p. 435).

3. Dyadic Analogs of Two Lebesgue Theorems

Let us remind the definition of dyadic convolution $f * g$ of two functions $f, g \in L[0,1)$:

$$(f * g)(x) = \int_{[0,1)} f(y) g(x \oplus y) dy, \quad x \in [0,1). \tag{6.11}$$

It is known that $f * g \in L(0,1)$ and $\widehat{(f * g)}(k) = \hat{f}(k)\hat{g}(k)$ where $\hat{f}$ are the Walsh - Fourier coefficients of the function f (see (6.5)).

The convolution (6.11) is well defined also for the case $f \in L^p(0,1)$, $1 \le p \le \infty$, $g \in L(0,1)$, and $f * g \in L^p[0,1)$.

Let us set

$$T_r^{(\alpha)}(x) = \sum_{k=0}^{2^r-1} k^\alpha w_k(x), \quad (\alpha \in R, r = 0, 1, \ldots). \tag{6.12}$$

In (6.12) and below we will assume $0^0 = 1$. We denote by $X[0,1)$ any of the spaces $L^p[0,1)$, $1 \le p \le \infty$.

He Zelin [8] introduced the following

DEFINITION 6.2 *Let $\alpha \in R$, $f \in X[0,1)$ and there exists a function $g \in X[0,1)$ such that*

$$\lim_{r\to\infty} \|f * T_r^{(\alpha)} - g\|_X = 0.$$

Then, for $\alpha > 0$ (or $\alpha < 0$) the function g is called the strong dyadic derivative (SDD) of order α (or the strong dyadic integral (SDI) of order $(-\alpha)$) of the function f in the space $X[0,1)$.

In both cases we will write $g = T^{(\alpha)} f$.

He Zelin [8] proved the equality $T^{(-1)} f = I(f)$, where $I(f)$ is the strong dyadic integral of the function $f \in X[0,1)$ in the space $X[0,1)$, which was introduced by P.L. Butzer and H.J. Wagner [1]. Moreover, in [8] for $\alpha \in R$, the equality

$$(\widehat{T^{(\alpha)} f})(k) = k^\alpha \hat{f}(k), \quad k \in Z_+, \quad \text{and} \quad f, T^{(\alpha)} f \in X[0,1),$$

has been proved.

For $\alpha > 0$ every function $f \in L^p[0,1)$ has the SDI $T^{(-\alpha)} f$ in the space $L^p[0,1)$ and

$$T^{(-\alpha)} f = (f * T_\infty^{(-\alpha)}),$$

where

$$T_\infty^{(\alpha)}(x) = \sum_{k=0}^{\infty} k^{-\alpha} w_k(x) \in L^p[0,1).$$

The series $\sum_{k=0}^{\infty} k^{-\alpha} w_k(x)$ converges to the function $T^{(-\alpha)} f$ in the space $L[0,1)$ and also at each point $x \in (0,1)$ if $\alpha > 0$.

It is proven in [8] that if $T^{(\alpha)} f$ exists in the space $L^p[0,1)$, $1 \le p \le \infty$ for some $\alpha \in R \setminus \{0\}$, then $T^{(-\alpha)}(T^{(\alpha)} f) = f$.

Let us introduce the pointwise analog of the Definition 6.2.

DEFINITION 6.3 *Let $\alpha \in R$, $f \in L[0,1)$ and there exists the finite limit*

$$t^{(\alpha)} f(x) \equiv \lim_{r\to\infty} (T_r^{(\alpha)} * f)(x)$$

at the point $x \in [0,1)$. Then, for $\alpha \ge 0$ (or $\alpha < 0$) the number $t^{(\alpha)} f(x)$ is called the dyadic derivative (DD) of order α (or the dyadic integral (DI) of order $(-\alpha)$ respectively) of the function f at the point x.

The following statement is true.

THEOREM 6.4 *Let $\alpha \in R \setminus \{0\}$ and the function $f \in L[0,1)$ has the SDD $T^{(\alpha)}f$ of order α in the space $L[0,1)$ for $\alpha > 0$ (or it has the SDI $T^{(\alpha)}f$ of order $(-\alpha)$ for $\alpha < 0$) in the space $L[0,1)$. Then, there exists $t^{(-\alpha)}(T^{(\alpha)}f)(x)$ at the point $x \in [0,1)$ if and only if x is a binary d-point of the function f. In this case,*

$$t^{(-\alpha)}(T^{(\alpha)}f)(x) = f(x).$$

In particular, the last equality is true almost everywhere on $[0,1)$.

It follows from this theorem the following corollary.

COROLLARY 6.1 *Under the conditions of the Theorem 6.4, the equality*

$$t^{(-\alpha)}(T^{(\alpha)}f)(x) = f(x)$$

holds at each dyadic Lebesgue point $x \in [0,1)$ of the function f. In particular, this equality is valid almost everywhere on $[0,1)$.

For $\alpha = -1$ this corollary may be considered as a dyadic analog of the classical Lebesgue theorem on the differentiation of indefinite Lebesgue integral. For $\alpha = 1$ it may be considered as a dyadic analog of the classical Lebesgue theorem on the integration of the ordinary derivative (see, for example, [16], Ch. 9).

Let us set

$$U_r^{(\alpha)}(x) = w_0(x) + \sum_{n=0}^{r} \sum_{j=0}^{2^n-1} 2^{n\alpha} w_{2^n+j}(x), \quad (\alpha \in R, r = 0, 1, \ldots).$$

DEFINITION 6.4 *Let $\alpha \in R$, $f \in X[0,1)$. If there exists a function $g \in X[0,1)$ such that*

$$\lim_{r\to\infty} \|f * U_r^{(\alpha)} - g\|_X = 0,$$

then for $\alpha > 0$ (or $\alpha < 0$) the function g is called the modified strong dyadic derivative (MSDD) of order α (or the modified strong dyadic integral (MSDI) of order $(-\alpha)$ respectively) of the function f in the space $X[0,1)$.

In this case we will write $g = U^{(\alpha)}f$.

DEFINITION 6.5 *Let $\alpha \in R$, $f \in L[0,1)$. If there exists the finite limit*

$$u^{(\alpha)}f(x) \equiv \lim_{r\to\infty} (U_r^{(\alpha)} * f)(x)$$

at the point $x \in [0,1)$, then for $\alpha > 0$ (or $\alpha < 0$) the number $u^{(\alpha)} f(x)$ is called the modified dyadic derivative (MDD) of order α (or the modified dyadic integral (MDI) of order $|\alpha|$ respectively) of the function f at the point x.

THEOREM 6.5 *Let $\alpha \in R \setminus \{0\}$ and for the function $f \in L[0,1)$ there exists $U^{(\alpha)} f$ in the space $L[0,1)$. Then there exists $u^{(\alpha)}(U^{(\alpha)} f)(x)$ at the point $x \in [0,1)$ if and only if x is a binary d-point of the function f. In this case*

$$u^{(-\alpha)}(U^{(\alpha)} f)(x) = f(x).$$

In particular this equality holds almost everywhere on $[0,1)$.

COROLLARY 6.2 *Let $\alpha \in R \setminus \{0\}$ and for the function $f \in L[0,1)$ there exists $U^{(\alpha)} f$ in the space $L[0,1)$. Then the equality*

$$u^{(-\alpha)}(U^{(\alpha)} f)(x) = f(x)$$

holds at each dyadic Lebesgue point $x \in [0,1)$ of the function f, and hence almost everywhere on $[0,1)$.

The results similar to the Corollary 6.2 for the functions $f \in L(R_+)$ were obtained by us in the paper [12] (see also [10], Ch. 3).

Let us note that for the functions $f \in L[0,1]$ the equality $(I_r(f))^{[r]}(x) = f(x)$ holds almost everywhere on $[0,1)$ if $\int_0^1 f(x)dx = 0$ and r is a natural number, where $I_r(f)$ is the strong dyadic integral and $f^{[r]}$ is the strong dyadic derivative of order r in the sense of Butzer and Wagner. That was proved by F. Shipp [17] as an answer on a question of H.J. Wagner [18]. But in the paper [17] there is not any characterization of the points in which the equality mentioned above holds.

Let us remind the notion of the dyadic convolution $f * g$ of two functions $f, g \in L(R_+)$

$$(f * g)(x) = \int_{R_+} f(y) g(x \oplus y) dy, \quad x \in R_+. \tag{6.13}$$

As for the ordinary convolution the following equality $(f \tilde{*} g) = \tilde{f}\tilde{g}$ holds. The dyadic convolution (6.13) is well defined also in the case $f \in L^p(R_+)$, $1 \le p \le \infty$, $g \in L(R_+)$, and $f * g \in L^p(R_+)$.

Let us introduce the notation

$$\Lambda_n^{\{\alpha\}}(x) = \int_0^{2^n} t^\alpha \psi(x,t) dt, \quad (x \in R_+, \alpha > 0, n \in Z).$$

This function is defined and bounded on R_+ for $\alpha > 0$ and $n \in Z$.

DEFINITION 6.6 *If $\alpha > 0$ and for the function $f \in L(R_+)$ there exists the finite limit*

$$d^{\{\alpha\}}(f)(x) = \lim_{n\to\infty}(f * \Lambda_n^{\{\alpha\}})(x)$$

at the point $x \in R_+$, then the number $d^{\{\alpha\}}(f)(x)$ is called the dyadic derivative (DD) of order α of the function f at the point x.

The following theorem is an analog of the Theorem 6.5 for functions defined on the positive half-line.

THEOREM 6.6 *Let $\alpha > 0$ and the function $f \in L(R_+)$ has the WSDI of order α in the space $L(R_+)$. Then the DD $d^{\{\alpha\}}(I_{\{\alpha\}}(f))(x)$ exists at the point $x \in R_+$ if and only if x is a binary d-point of the function f. In this case the equality $d^{\{\alpha\}}(I_{\{\alpha\}}(f))(x) = f(x)$ is valid. In particular, this equality holds almost everywhere on R_+.*

COROLLARY 6.3 *Let $\alpha \in R\setminus\{0\}$ and the function $f \in L(R_+)$ has the WSDI $I_{\{\alpha\}}(f)$ in the space $L(R_+)$. Then, the equality $d^{\{\alpha\}}(I_{\{\alpha\}}(f))(x) = f(x)$ holds at each dyadic Lebesgue point $x \in [0,1)$ of the function f, and hence almost everywhere on* $[0,1)$. *Therefore this equality is valid almost everywhere on the segment* $[0,1)$.

Let us note that the equality $(I_{\{1\}}(f))^{\{1\}}(x) = f(x)$ was proved by J. Pal and F. Shipp almost everywhere on R_+ (see [19] or [11], p. 445). But they used an other definition of the derivative, namely that one of P.L. Butzer and H.J. Wagner.

4. Dyadic Integration by Parts

The results of this section can be considered as the fractional dyadic analogs of classical formula of integration by parts.

THEOREM 6.7 *Let $f \in L^p[0,1)$, $g \in L^q[0,1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < \infty$ and for some $\alpha > 0$ the SDD $T^{(\alpha)}f$ and $T^{(\alpha)}g$ exist in the spaces $L^p[0,1)$ and $L^q[0,1)$ respectively. Then the equality*

$$\int_0^1 (T^{(\alpha)}f)(x)g(x)dx = \int_0^1 f(x)(T^{(\alpha)}g)(x)dx \tag{6.14}$$

holds. For $\alpha < 0$ the similar statement is valid also for the strong dyadic integrals $T^{(\alpha)}f$ and $T^{(\alpha)}g$.

An analog of this theorem is true also for the modified strong dyadic derivatives and integrals.

THEOREM 6.8 *Let $f \in L^p[0,1)$, $g \in L^q[0,1)$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \le p < \infty$, and for some $\alpha > 0$ the MSDD $U^{(\alpha)}f$ and $U^{(\alpha)}g$ exist in the spaces $L^p[0,1)$ and $L^q[0,1)$ respectively. Then the equality*

$$\int_0^1 (U^{(\alpha)}f)(x)g(x)dx = \int_0^1 f(x)(U^{(\alpha)}g)(x)dx,$$

holds. For $\alpha < 0$ the similar statement is valid also for the MSDI $U^{(\alpha)}f$ and $U^{(\alpha)}g$.

Let us formulate the similar results for the functions defined on the positive half-line R_+. To this aim we introduce the function $h : (0,+\infty) \to (0,+\infty)$ as follows

$$h(x) = 2^{-n}, \quad 2^n \le x < 2^{n+1}, \quad n \in Z. \tag{6.15}$$

For $\alpha > 0$ we set

$$\Lambda_n^\alpha(x) = \int_0^{2^n} (h(t))^{-\alpha}\psi(x,t)dt, \quad x \in R_+.$$

LEMMA 6.2 *For $\alpha > 0$, $n \in Z$, the function Λ_n^α is defined and bounded on the positive half-line R_+ and $\Lambda_n^\alpha \in L(R_+)$. Moreover,*

$$\tilde{\Lambda}_n^\alpha(x) = (h(x))^{-\alpha}X_{[0,2^n)}(x) \quad \textit{for} \quad x > 0 \quad \textit{and} \quad \tilde{\Lambda}_n^\alpha(0) = 0,$$

where X_E is the characteristic function of the set $E \subset R_+$ (see [10], Ch. 3).

Using this lemma we can introduce the following definition.

DEFINITION 6.7 *If for the function $f \in L^p(R_+)$, $1 \le p \le \infty$, there exists a function $\varphi \in L^p(R_+)$ such that*

$$\lim_{n\to+\infty} \|f * \Lambda_n^\alpha - \varphi\|_{L^p(R_+)} = 0,$$

then the function $\varphi \equiv D^{(\alpha)}f$ is called the modified strong dyadic derivative (MSDD) of order α of the function f in the space $L^p(R_+)$.

THEOREM 6.9 *Let $f \in L^p(R_+)$, $g \in L^q(R_+)$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \le p < \infty$ and for some $\alpha > 0$ the MSDD $D^\alpha f$ and $D^\alpha(g)$ exist in the spaces $L^p(R_+)$ and $L^q(R_+)$ respectively. Then the equality*

$$\int_{R_+} D^\alpha f(x)g(x)dx = \int_{R_+} f(x)D^\alpha g(x)dx$$

is valid.

LEMMA 6.3 *For $\alpha > 0$ and $n \in Z$, the function*

$$W_n^\alpha(x) \equiv \lim_{m\to\infty} \int_{2^{-n}}^{2^m} \psi(x,y)(h(y))^\alpha dy$$

at each point $x > 0$ is well defined, $W_n^\alpha \in L(R_+)$ and $W_n^\alpha(x) = 0$ for $x \geq 2^n$. (See [20] or [10], Ch. 3).

Using this lemma we can introduce the following definition.

DEFINITION 6.8 *Let $\alpha > 0$, $f, g \in L^p(R_+)$, $1 \leq p \leq \infty$, and*

$$\lim_{n\to+\infty} \|f * W_n^\alpha - g\|_{L^p(R_+)} = 0.$$

Then, the function $g \equiv J_\alpha f$ is called the modified strong dyadic integral (MSDI) of order α of the function f in the space $L^p(R_+)$.

THEOREM 6.10 *Let $f \in L^p(R_+)$, $g \in L^q(R_+)$ $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p < \infty$, $1 \leq q > \infty$, and for some $\alpha > 1$, the MSDI $J_\alpha f$ and $J_\alpha g$ exist in the spaces $L^p(R_+)$ and $L^q(R_+)$ respectively. Then the equality*

$$\int_{R_+} J_\alpha f(x) g(x) dx = \int_{R_+} f(x) J_\alpha g(x) dx$$

holds.

An analog of this theorem is valid also for the Wagner strong dyadic integrals.

THEOREM 6.11 *Let $f \in L^p(R_+)$, $g \in L(R_+)$, $frac1p + \frac{1}{q} = 1$, $1 \leq p < \infty$, $1 \leq q < \infty$, and for some $\alpha > 0$ the WSDI $I_{\{\alpha\}} f$ and $I_{\{\alpha\}} g$ exist in the spaces $L^p(R_+)$ and $L^q(R_+)$ respectively. Then, the equality*

$$\int_{R_+} I_{\{\alpha\}} f(x) g(x) dx = \int_{R_+} f(x) I_{\{\alpha\}} g(x) dx$$

holds.

5. Fractional Dyadic Integration and Differentiation of an Integral by a Parameter

Let us remind that according to the Lemma 6.2 for $\alpha > 0$ and $n \in Z_+$ the inclusion

$$\Lambda_n^\alpha \in L(R_+) \cap L^\infty(R_+)$$

is valid. Thus, we can introduce the following definition.

DEFINITION 6.9 *Let $\alpha > 0$ and for the function $f \in L(R_+) \cup L^\infty(R_+)$ there exists finite limit*

$$d^{(\alpha)}(f)(x) = \lim_{n\to+\infty}(f * \Lambda_n^\alpha)(x)$$

at the point $x \in R_+$.

Then the number $d^{(\alpha)}(f)(x) = (f)(x)$ is called the modified dyadic derivative (MDD) of order α of the function f at the point x.

Let us introduce the notation

$$d_n^{(\alpha)} f(x) = (f * \Lambda_n^\alpha)(x),$$

where $n \in Z_+$, $\alpha > 0$. Then the existence of MDD $d^{(\alpha)} f(x)$ of the function $f \in L(R_+) \cup L^\infty(R_+)$ at the point $x \in R_+$ can be written in the form

$$\lim_{n\to+\infty} d_n^{(\alpha)} f(x) = d^{(\alpha)} f(x).$$

Below for the function of two variables $f(x,t)$ the notation $d^{(\alpha)} f(x_0, t)$ denotes the MDD of order α at the point $x_0 \in R_+$ by the first argument.

Let us consider the problem of dyadic differentiation of the integral

$$F(x) = \int_E f(x,t)dt, \tag{6.16}$$

where E is a Lebesgue measurable set from R^m, $m \in N$ and $x \in R_+$.

THEOREM 6.12 *Let the Lebesgue measurable function $f : R_+ \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x,t)| \le \varphi(t)$ for all $x \in R_+$ and almost all $t \in E$. We also assume that there exists the MDD $d^{(\alpha)} f(x_0, t)$ at the point $x_0 \in R_+$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let the integral $\int_E d^{(\alpha)} f(x_0,t)dt$ converges and the sequence $d_n^{(\alpha)} f(x_0, t)$ has an integrable majorant on the set E. Then, the function (6.16) has the MDD of order α at the point x_0 and the equality*

$$d^{(\alpha)} F(x_0) = \int_E d^{(\alpha)} f(x_0,t)dt$$

is valid.

COROLLARY 6.4 *Let $\alpha > 0$, $\varphi \in L(R_+)$, and $h^{-\alpha}\varphi \in L(R)$. Then the Walsh-Fourier transform $\tilde{\varphi}$ of the function φ has the MDD $d^{(\alpha)}(\tilde{\varphi})(x)$ of order α at each point $x \in R_+$ and the equality*

$$d^{(\alpha)}(\tilde{\varphi})(x) = (\widetilde{h^{-\alpha}\varphi})(x)$$

holds.

The proof of this corollary is based on the Theorem 6.12 and the following lemma.

LEMMA 6.4 *The generalized Walsh function $\psi(\circ, y) \equiv \psi_y(\circ)$ has the MDD of order $\alpha > 0$ at each point $x \in R_+$. Moreover, $d^{(\alpha)}(\psi_0)(x) \equiv 0$ on R_+ and for $y > 0$ the equality*

$$d^{(\alpha)}(\psi_y)(x) \equiv (h(y))^{-\alpha}\psi_y(x)$$

is valid (see [10] , Ch. 3).

Let us note that the Lemma 6.4 was proved at first in our paper [20] and J. Pal [21] proved a similar result for pointwise derivative of first order of Butzer-Wagner type.

Now we consider the problem of fractional dyadic integration of the function (6.16). Taking into account the Lemma 6.3 we introduce the following definition.

DEFINITION 6.10 *Let $\alpha > 0$ and for the function $f \in L^\infty(R_+)$ there exists the finite limit*

$$j_\alpha(f)(x) = \lim_{n\to+\infty}(f * W_n^\alpha)(x)$$

at the point x. Then the number $j_\alpha(f)(x)$ is called the modified dyadic integral (MDI) of order α of the function f at the point x.

For the function $f \in L^\infty(R_+)$ we set

$$j_n^{(\alpha)} f(x) = (f * W_n^\alpha)(x)$$

where $n \in Z_+$, $\alpha > 0$. Then the existence of the MDI $j_n^\alpha f(x)$ for the function $f \in L^\infty(R_+)$ at the point $x \in R_+$ can be written in the form $\lim_{n\to+\infty} j_n^{(\alpha)} f(x) = j_\alpha f(x)$.

Below for the function of two variables $f(x,t)$ the symbol $j_\alpha f(x_0,t)$ denotes the MDI of order α at the point $x_0 \in R_+$ by the first argument.

THEOREM 6.13 *Let the Lebesgue measurable function $f : R_+ \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x,t)| \leq \varphi(t)$ for all $x \in R_+$ and almost all $t \in E$. We also assume that there exists the MDI $j_\alpha f(x_0,t)$ at the point $x_0 \in R_+$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let the integral $\int_E j_\alpha f(x_0,t)dt$ converges and the sequence $j_n^{(\alpha)} f(x_0,t)$ has an* *integrable majorant on the set E. Then, the function (6.16) has the MDI of order α at the point x_0 and the equality*

$$j_\alpha F(x_0) = \int_E j_\alpha f(x_0, t) dt,$$

is valid.

COROLLARY 6.5 *Let $\alpha > 0$, $\varphi \in L(R_+)$ and $h^\alpha \varphi \in L(R_+)$. Then, the Walsh-Fourier transform $\tilde{\varphi}$ of the function φ has the MDI $j_\alpha(\tilde{\varphi})(x)$ of order α at each point x and the equality*

$$j_\alpha(\tilde{\varphi})(x) = (h^{\tilde{\alpha}} \varphi)(x)$$

holds.

The proof of this corollary is based on the Theorem 6.13 and the following lemma.

LEMMA 6.5 *The generalized Walsh function $\psi(x, y) \equiv \psi_y(x)$ has the MDI of order $\alpha > 0$ at each point $x \in R_+$. Moreover, $j_\alpha(\psi_0)(x) \equiv 0$ on R_+ and for $y > 0$, the equality*

$$j_\alpha(\psi_y)(x) \equiv (h(y))^\alpha \psi_y(x)$$

is valid (see [20] or [10], Ch. 3).

Let us set

$$t_n^{(\alpha)} f(x_0, y) \equiv (T_n^{(\alpha)} * f(\circ, y))(x_0),$$

where the kernel $T_n^{(\alpha)}$ was introduced in (6.12).

THEOREM 6.14 *Let the Lebesgue measurable function $f : [0, 1) \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x, y)| \leq \varphi$ for all $x \in [0, 1)$ and almost all $y \in E$. We also assume that there exists the MDD $t^{(\alpha)} f(x_0, y)$ for some $\alpha > 0$ (or the MDI $t^{(\alpha)} f(x_0, y)$ for some $\alpha < 0$) at the point $x_0 \in [0, 1)$ for almost all $y \in E$. Moreover, let the integral $\int_E t^{(\alpha)} f(x_0, y) dy$ converges and the sequence $t_n^{(\alpha)} f(x_0, y)$ has an integrable majorant on the set E. Then, the function (6.16) has the DD $t^{(\alpha)} F(x_0)$ of order $\alpha > 0$ (or the DI $t^{(\alpha)} F(x_0)$ of order $(-\alpha)$) at the point x_0 and the equality*

$$t^{(\alpha)} F(x_0) = \int_E t^{8\alpha)} f(x_0, y) dy$$

is valid.

Using the kernel $W_n^{\{\alpha\}}(x)$ defined in Lemma 6.1, we introduce the following definition.

DEFINITION 6.11 *If $\alpha > 0$ and there exists the finite limit*

$$i_{\{\alpha\}}(f)(x) = \lim_{n\to+\infty}(f * W_n^{\{\alpha\}})(x)$$

for the function $f \in L^\infty(R_+)$ at the point $x \in R_+$, then the number $i_{\{\alpha\}}(f)(x)$ is called the Butzer-Wagner dyadic integral (BWDI) of order α of the function f at the point x.

Let us set

$$i_n^{\{\alpha\}} f(x,t) = \int_{R_+} f(y,t) W_n^{\{\alpha\}}(x \oplus y) dy, \quad x \in R_+, t \in E. \tag{6.17}$$

THEOREM 6.15 *Let the Lebesgue measurable function $f : R_+ \times E \to R$ has a majorant $\varphi \in L(E)$ such that $|f(x,t)| \leq \varphi(t)$ for all $x \in R_+$ and $t \in E$. We also assume that there exists BWDI $i_{\{\alpha\}} f(x_0,t)$ at the point $x_0 \in R_+$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let the integral $\int_E i_{\{\alpha\}} f(x_0,t)dt$ converges and the sequence $i_n^{\{\alpha\}} f(x_0,t)$ has an integrable majorant on the set E. Then, the function (6.16) has the BWDI $i_{\{\alpha\}} F(x_0)$ of order α at the point x_0 and the equality*

$$i_{\{\alpha\}} F(x_0) = \int_E i_{\{\alpha\}} f(x_0,t)dt$$

is valid.

Now let us consider the problem of modified strong dyadic differentiation and integration of the function (6.16). We will use the notation

$$d_n^\alpha f(x,t) = \int_{R_+} f(y,t)\Lambda_n^\alpha(x \oplus y)dy, \quad x \in R_+, t \in E, \alpha > 0.$$

THEOREM 6.16 *Let the measurable function $f : R_+ \times E \to R$ is such that*

$$\int_{R_+} \left\{ \int_E |f(x,t)| dt \right\} dx < \infty, \tag{6.18}$$

and there exists the MSDD $D^\alpha f(x,t)$ by the parameter x in the space $L(R_+)$ for almost all $t \in E$ and some $\alpha > 0$. Moreover, let

$$\lim_{n\to\infty} \int_E \|d_n^\alpha f(\circ,t) - D^\alpha f(\circ,t)\|_{L(R_+} dt = 0.$$

Then the function (6.16) has the MSDD $D^{\alpha}F$ of order α in the space $L(R_+)$ and

$$D^{\alpha}F(\circ) = \int_E D^{\alpha}f(\circ,t)dt.$$

Let us introduce the notation

$$j_n^{\alpha}f(x,t) = \int_{R_+} f(y,t)W_n^{\alpha}(x \oplus y)dy, \quad x \in R_+, t \in E.$$

THEOREM 6.17 *Let the measurable function $f : R_+ \times E \to R$ satisfies the condition (6.18). Moreover, let the function $f(x,t)$ has MSDI $J_{\alpha}f(\circ,t)$ of some order $\alpha > 0$ in the space $L(R_+)$ for almost all $t \in E$ and*

$$\lim_{n\to\infty} \int_E \|j_n^{\alpha}f(\circ,t) - J_{\alpha}f(\circ,t)\|_{L(R_+)}dt = 0.$$

Then the function (6.16) has MSDI $J_{\alpha}F$ of order α in the space $L(R_+)$ and

$$J_{\alpha}f(\circ) = \int_E J_{\alpha}f(\circ,t)dt.$$

The similar result is valid also for the Wagner strong dyadic integral.

THEOREM 6.18 *Let the measurable function $f : R_+ \times E \to R$ satisfies the condition (6.18). Moreover, let the function $f(x,t)$ has the WSDI $I_{\{\alpha\}}f(\circ,t)$ of some order $\alpha > 0$ in the space $L(R_+)$ for almost all $t \in E$ and*

$$\lim_{n\to\infty} \int_E \|i_n^{\{\alpha\}}f(\circ,t) - I_{\{\alpha\}}f(\circ,t)\|_{L(R_+)}dt = 0.$$

where $i_n^{\{\alpha\}}f(x,t)$ is defined in (6.17). Then the function (6.16) has the WSDI $I_{\{\alpha\}}F$ of order α in the space $L(R_+)$ and the equality

$$I_{\{\alpha\}}F(\circ) = \int_E I_{\{\alpha\}}f(\circ,t)dt$$

holds.

Let us introduce the notation $t_r^{(\alpha)}f(\circ,t) = T_r^{(\alpha)} * f(\circ,t)$, where the kernel $T_r^{(\alpha)}$ is defined by the equality (6.12).

THEOREM 6.19 *Let the measurable function $f : R_+ \times E \to R$ satisfies the condition*

$$\int_{[0,1)} \{|f(x,t)|dt\}\, dx < \infty.$$

Moreover, let the function $f(x,t)$ has the SDD $T^{(\alpha)}f(\circ,t)$ of some order $\alpha > 0$ (or it has the SDI $T^{(\alpha)}f(\circ,t)$ if $\alpha < 0$) for almost all $t \in E$ and

$$\lim_{n\to\infty} \int_E \|t_n^\alpha f(\circ,t) - T^{(\alpha)} f(\circ,t)\|_{L[0,1)} dt = 0.$$

Then the function (6.16) has the SDD $T^{(\alpha)}F$, if $\alpha > 0$ (or it has the SDI $T^{(\alpha)}F$, if $\alpha < 0$) in the space $L[0,1)$ and the equality

$$T^{(\alpha)}F(\circ) = \int_E T^{(\alpha)} f(\circ,t) dt$$

is valid.

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Chapter 7

WAVELET LIKE TRANSFORM ON THE BLASCHKE GROUP

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Abstract Blaschke-functions play an important role in system identification. These functions form a group with respect to the composition of functions. In the papers [5],[6],[7] we introduced a new transform connected to this group. It is in same relation with this group as the affine wavelet transform with the affine group, or the Gábor-transform with the Heisenberg-group. In this paper we give a summary of the main concepts and results. The first section contains the basic notations, definitions and results connected to the representations and the voice transform.

In section 2 the voice transform of the Blaschke group is studied on the Lebesgue $L^2(\mathbf{T})$ space and on the Hardy space $H^2(\mathbb{D})$. The discrete Laguerre system can be obtained as wavelet system with respect to the Blaschke group. In section 3 we consider the voice transform on the Bergman space.

1. The Voice Transform

In signal processing and image reconstruction the Fourier-, wavelet-, Gábor-transforms play important roles. There exists a common generalization of these transformations, the so-called *Voice-transformation*. In this section we summarize the basic notions used in the definition of Voice-transform. We also present the definition and the most important properties of this transform.

In the construction of Voice-transform the starting point will be a locally compact topological group $(G, \cdot)$. It is known that every locally compact topological group has nontrivial left- and right-translation invariant Borel-measures,

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called left invariant and right invariant *Haar-measures.* Let m be a nontrivial left-invariant Haar measure of G, and let $f : G \to \mathbb{C}$ be a Borel-measurable function integrable with respect to the m. The integral of f will be denoted by $\int_G f\,dm = \int_G f(x)\,dm(x)$. Because of the left-translation invariance of the measure m it follows that

$$\int_G f(x)\,dm(x) = \int_G f(a^{-1}\cdot x)\,dm(x) \quad (a \in G).$$

There exist groups whose left invariant Haar measure is no right invariant. If the left invariant Haar measure of G is in the same time right invariant then G is called *unimodular group*. On a given group, Haar measure is unique only up to constant multiples. It is trivial that the commutative groups are unimodular. Furthermore it can be proved that if the left Haar measure is invariant under the inverse transformation $G \ni x \to x^{-1} \in G$, then G is also unimodular (for details see [8],[9]).

In the definition of the voice-transform a *unitary representation of the group* $(G, \cdot)$ is used. Let us consider a Hilbert-space $(H, \langle\cdot,\cdot\rangle)$ and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear bijections which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group, the neutral element of which is I, the identity operator on H. The inverse element of $U \in \mathcal{U}$ is the operator U^{-1}. It is equal to the adjoint operator U^*. The homomorphism of the group $(G, \cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying

$$\begin{aligned} &\text{i)} \quad U_{x\cdot y} = U_x \circ U_y \quad (x, y \in G), \\ &\text{ii)} \quad G \ni x \to U_x f \in H \ \text{ is continuous for all } f \in H \end{aligned} \tag{7.1}$$

is called the unitary representation of $(G, \cdot)$ on H. The *voice transform* of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex-valued) function on G defined by

$$(V_\rho f)(x) := \langle f, U_x\rho\rangle \quad (x \in G, f, \rho \in H). \tag{7.2}$$

For any representation $U : G \to \mathcal{U}$ and for each $f, \rho \in H$ the voice transform $V_\rho f$ is a continuous and bounded function on G.

The set of continuous bounded functions defined on the group G with the usual norm $\|F\| := \sup\{|F(x)| : x \in G\}$, form a Banach space. From the unitarity of $U_x : H \to H$ follows that, for all $x \in G$

$$|(V_\rho f)(x)| = |\langle f, U_x\rho\rangle| \le \|f\|\|U_x\rho\| = \|f\|\|\rho\|,$$

consequently V_ρ is linear, bounded and $\|V_\rho\| \le \|\rho\|$.

Taking as starting point (not necessarily commutative) locally compact groups and a unitary representation we can construct important transformations in signal processing and control theory in this way. For example the Fourier transform, the affine wavelet transform and the Gábor-transform are all special voice transforms (see [4], [8]).

The invertibility of the voice transform V_ρ is connected to the irreducibility of the representation U. The representation U is called *irreducible* if the only closed invariant subspaces of H, i.e. the closed subspaces H_0 which satisfy $U_x H_0 \subset H_0$ $(x \in G)$, are $\{0\}$ and H. Since the closure of the linear span of the set

$$\{U_x \rho : x \in G\} \tag{7.3}$$

is always a closed invariant subspace of H, it follows that U is irreducible if and only if the collection (1.3) is a closed system for any $\rho \in H, \rho \neq 0$.

The property of irreducibility gives a simple criterion for deciding when a voice transform is $1-1$ (see [4], [8]):

I. *A voice transform V_ρ generated by a unitary representation U is $1-1$ for all $\rho \in H \setminus \{0\}$ if and only if U is irreducible.*

The function $V_\rho f$ is continuous on G but in general is not square integrable. If there exists $\rho \in H, \rho \neq 0$ such that $V_\rho \rho \in L^2_m(G)$, then the representation U is called *square integrable* and ρ is called admissible for U. For a fixed square integrable U the collection of admissible elements of H will be denoted by H^*. Choosing a convenient $\rho \in H^*$ the voice transform $V_\rho : H \to L^2_m(G)$ will be unitary. This is consequence of the following claim (see [4], [8]):

II. *Let $(U_a)_{a\in G}$ be an irreducible square integrable representation of the group G on the Hilbert space H. The collection H^* of admissible elements is a linear subspace of H and for every $\rho \in H^*$ the voice transform of the function f is square integrable on G, namely $V_\rho f \in L^2_m(G)$, if $f \in H$. Moreover there is a symmetric, positive bilinear map $B : H^* \times H^* \to \mathbb{R}$ such that*

$$[V_{\rho_1} f, V_{\rho_2} g] = B(\rho_1, \rho_2)\langle f, g\rangle \quad (\rho_1, \rho_2 \in H^*, f, g \in H), \tag{7.4}$$

where $[\cdot,\cdot]$ is the usual inner product in $L^2_m(G)$. If the group G is unimodular then $B(\rho, \rho) = c\langle \rho, \rho\rangle$ $(\rho \in H^*)$, where $c > 0$ is a constant. In this case if we choose ρ so that $\langle \rho, \rho\rangle = 1/c$ then

$$[V_\rho f, V_\rho g] = \langle f, g\rangle \quad (f, g \in H). \tag{7.5}$$

In the next sections we will construct voice transform using so called *multiplier representations* generated by a collection of multiplier functions defined in the following way: $F_a : G \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ $(a \in G)$ is a *collection of multiplier functions* if

$$F_e = 1, \;\; F_{a_1 \cdot a_2}(x) = F_{a_1}(a_2 \cdot x) F_{a_2}(x) \;\; (a_1, a_2, x \in G), \tag{7.6}$$

where e is the neutral element of G. It can be proved that

$$(U_a f)(x) := F_{a^{-1}}(x) f(a^{-1} \cdot x) \;\; (a, x \in G)$$

satisfies

$$U_{a_1}(U_{a_2} f) = U_{a_1 \cdot a_2} f \quad (a_1, a_2 \in G),$$

so it is a representation of G on the space of all complex valued functions on G. If F_a is bounded and continuous on G for every $a \in G$, then $L^2_m(G)$ is an invariant subspace and $(U_a)_{a \in G}$ is a representation on G. The representations obtained as bellow are named *multiplier representations* (see [9]). Let us denote by L_a the left translation by a^{-1}, i.e., for any function $f : G \to \mathbb{C}$ set

$$(L_a f)(x) := f(a^{-1} \cdot x). \tag{7.7}$$

It can be proved that the voice transform and the left translation operator satisfies the following properties (see [4], [8]):

III. *Let $(U_a)_{a \in G}$ be a unitary multiplier representation of G which is generated by $F_a \in C(G) \cap L^\infty_m(G)$ $(a \in \mathbb{G})$. Then*

$$\begin{aligned}
&\text{i)} \quad V_\rho \circ U_a = L_a \circ V_\rho, \\
&\text{ii)} \quad V_\rho \circ L_a = L_a \circ V_\rho \circ M_a \\
&\text{iii)} \quad (V_\rho f)(x) = (V_f \rho)(x^{-1}) \quad (a, x \in G, \rho \in H),
\end{aligned}$$

where M_a denotes the multiplication by $F_{a^{-1}}$.

We note that i) and iii) are valid for all unitary representations.

2. The Voice Transforms on the Blaschke Group

The affine wavelet transform is a voice transform of the affine group which is a subgroup of the Möbius group (i.e. the group of linear fractional transformations with the composition operation). In this section we will study the voice transforms of another subgroup of the Möbius group, namely the voice transforms of the Blaschke group.

The Blaschke group

The so called *Blaschke functions* are defined as

$$B_a(z) := \varepsilon \frac{z - b}{1 - \bar{b} z} \;\; (z \in \mathbb{C}, a = (b, \varepsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}), \tag{7.8}$$

where

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \;\; \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}. \tag{7.9}$$

If $a \in \mathbb{B}$, then B_a is an 1-1 map on $\mathbb{T}$ and $\mathbb{D}$ a respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_a, a \in \mathbb{B}\}, \circ)$.

If we use the notations $a_j := (b_j, \varepsilon_j),\ j \in \{1, 2\}$ and $a := (b, \varepsilon) =: a_1 \circ a_2$ then

$$\begin{aligned} b &= \frac{b_1\bar{\varepsilon}_2 + b_2}{1 + b_1\bar{b}_2\bar{\varepsilon}_2} = B_{(-b_2,1)}(b_1\bar{\varepsilon}_2), \\ \varepsilon &= \varepsilon_1 \frac{\varepsilon_2 + b_1\bar{b}_2}{1 + \varepsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2,\varepsilon_1)}(\varepsilon_2). \end{aligned} \tag{7.10}$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \varepsilon) \in \mathbb{B}$ is $a^{-1} = (-b\varepsilon, \bar{\varepsilon})$.

Since $B_a : \mathbb{T} \to \mathbb{T}$ is a bijection it follows the existence of $\beta_a > \mathbb{R} \to \mathbb{R}$ such that $B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R})$, where β_a can be expressed in a explicit form. Namely, let us introduce the function

$$\gamma_r(t) := \int_0^t \frac{1 - r^2}{1 - 2rcoss + r^2}\, ds \ \ (t \in \mathbb{R}, 0 \le q < r). \tag{7.11}$$

Then

$$\begin{aligned} \beta_a(t) &:= \theta + \varphi + \gamma_{s(r)}(t - \varphi), \\ s &:= s(r) := \frac{1 + r}{1 - r} \ \ (a = (re^{i\varphi}, e^{i\theta}) \in \mathbb{B}, t \in \mathbb{R}). \end{aligned} \tag{7.12}$$

The group $(\mathbb{B}, \cdot)$ is unimodular and the integral of the function $f : \mathbb{B} \to \mathbb{C}$, with respect to the Haar-measure m of the group $(\mathbb{B}, \circ)$ can be written in the form

$$\int_{\mathbb{B}} f(a)\, dm(a) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2}\, db_1 db_2 dt, \tag{7.13}$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

We will study the voice transform of the Blaschke-group. In the construction there will be used a class of unitary representations of the Blaschke-group on the Hilbert space $H = L^2(\mathbb{T})$.

The voice transform on $L^2(\mathbb{T})$

In this section the voice transform on the Hilbert space $H = L^2(\mathbb{T})$ will be constructed, where the inner product is given by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{I}} f(e^{it})\overline{g(e^{it})}\,dt \ \ (f, g \in H).$$

The trigonometric system $\varepsilon_n(t) = e^{int}(t \in \mathbb{I}, n \in \mathbb{Z})$ is orthonormal and complete with respect this scalar product.

It can be proved that the function

$$F_a(e^{it}) := \sqrt{\beta_a'(t)}e^{i(\beta_a(t)-t)/2} \ (a \in \mathbb{B}, t \in \mathbb{I})$$

is a multiplier. The representation of the Blaschke-group on $L^2(\mathbb{T})$ generated by this multiplier is given by

$$\begin{aligned}(U_a f)(e^{it}) &:= F_{a^{-1}}(e^{it}) \cdot f \circ \beta_{a^{-1}}(t) \\ &= f(e^{i\beta_{a^{-1}}(t)})(\beta'_{a^{-1}}(t))^{1/2}e^{i(\beta_{a^{-1}}(t)-t)/2} \ \ (a \in \mathbb{B}).\end{aligned} \tag{7.14}$$

It can be proved (see [5], [6])

IV. *The representation U_a $(a \in \mathbb{B})$ given by the formula (7.14) is a unitary representation of the Blaschke-group on* $L^2(\mathbb{T})$. Denote by $H^2(\mathbb{T})$ the closure in $L^2(\mathbb{T})$-norm of the set span$\{\varepsilon_n, n \in \mathbb{N}\}$. The functions which belong to $H^2(\mathbb{T})$ can be obtained as boundary limits of the functions from Hardy-space $H^2(\mathbb{D})$ (see [3]). The restriction of the representation (7.14) on $H^2(\mathbb{T})$ can be considered as a representation on the Hilbert-space $H^2(\mathbb{D})$ and can be given in the form

$$\begin{aligned}(U_{a^{-1}}f)(z) &:= \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\bar{b}z)} f\Big(\frac{e^{i\theta}(z-b)}{1-\bar{b}z}\Big) \\ &\qquad \big(z = e^{it} \in \mathbb{T}, a = (b, e^{i\theta}) \in \mathbb{B}\big).\end{aligned} \tag{7.15}$$

The voice transform generated by $(U_a)_{a\in\mathbb{B}}$ is given by the following formulae

$$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle \ \ (f, \rho \in H^2(\mathbb{T})). \tag{7.16}$$

Let consider the shift operator

$$(S\varphi)(z) = z\varphi(z), \quad \varphi \in H^2(\mathbb{T}),$$

and let $\varphi = 1 \in H^2(\mathbb{T})$ be the *mother wavelet* (compare [2], [8]). Then the *discrete Laguerre* functions can be generated by the shift operator and by the representation operator in the following way (see [1]) :

$$\begin{aligned}\varphi_{a,m}(z) &:= (U_{a^{-1}}S^m\varphi)(z) \\ &= U_{a^{-1}}\varepsilon_m = \frac{\sqrt{\varepsilon(1-|b|^2)}}{(1-\bar{b}z)}\left(\frac{\varepsilon(z-b)}{1-\bar{b}z}\right)^m.\end{aligned} \tag{7.17}$$

It is known that $(\varphi_{a,m}, m \in \mathbb{N})$ forms an orthogonal basis in $L^2(\mathbb{T})$, for all $a \in \mathbb{B}$ (see [1]).

Let $V_{\varepsilon_m}f(a^{-1}) = \langle f, U_{a^{-1}}\varepsilon_m\rangle$ and let the projection operator Pf defined as

$$Pf(a,z) := \sum_{m=0}^{\infty} V_{\varepsilon_m}f(a^{-1})U_{a^{-1}}\varepsilon_m(z) \;\; (a \in \mathbb{B},\, z \in \mathbb{D}). \tag{7.18}$$

V. *For every $f \in H^2(\mathbb{T})$, $z = r_1e^{it} \in \mathbb{D}$, and $a \in \mathbb{B}$*

$$\lim_{r_1 \to 1} Pf(a,z) = f(e^{it}) \tag{7.19}$$

for ae. $t \in \mathbb{I}$ and in H^2 norm. If $f \in C(\mathbb{T})$, then the convergence is uniform (see [5]).

If $a = e = (0,1)$ then $V_{\varepsilon_m}f(e) = \langle f, \varepsilon_m\rangle$ and $U_{e^{-1}}\varepsilon_m(z) = \varepsilon_m$. Consequently in this special case (7.19) is the analogue of Abel summation for the trigonometric series.

3. The Voice Transform on the Bergman Space

In this section we introduce the voice transform by using a class of unitary representations of the Blaschke group on *Bergman spaces* (see [3]).

Let the set of analytic function $f : \mathbb{D} \to \mathbb{C}$ denoted by $\mathcal{A}$. For $m \in \mathbb{N}, m \geq 2$ let us consider the following subset of analytic functions:

$$\mathcal{B}^m(\mathbb{D}) := \{f \in \mathcal{A} : \frac{1}{2\pi}\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{m-2}\,dxdy < \infty \;\; (z = x+iy \in \mathbb{D})\}.$$

The set $\mathcal{B}^m(\mathbb{D})$ with the scalar product

$$\langle f, g\rangle_m := \frac{1}{2\pi}\int_{\mathbb{D}} f(z)\overline{g(z(z)}\,(1-|z|^2)^{m-2}\,dxdy \;\; (z = x+iy \in \mathbb{D})$$

is a Hilbert space. In the special case when $m = 2$, $\mathcal{B}^2(\mathbb{D})$ is the *Bergman space.* It can be proved that the function

$$f(z) := \sum_{n=0}^{\infty} c_n z^n \;\; (z \in \mathbb{D})$$

from $\mathcal{A}$ belongs to the set $\mathcal{B}^m(\mathbb{D})$ if and only if the coefficients of f satisfy

$$\sum_{n=0}^{\infty} |c_n|^2 \lambda_n^{[m]} < \infty,$$

where

$$\lambda_n^{[m]} := \int_0^1 (1-r^2)^{m-2} r^{2n+1}\, dr \ \ (m \geq 2, n \in \mathbb{N}).$$

When $m = 2$, then

$$\lambda_n^{[2]} := \frac{1}{2(n+1)} \ \ (n \in \mathbb{N}).$$

The Hardy space $H^2(\mathbb{D})$ is a subspace of the Bergman-space $\mathcal{B}^2(\mathbb{D})$.

Let us consider the collection of functions

$$F_a(z) := \frac{\sqrt{\varepsilon(1-|b|^2)}}{1-\overline{b}\, z} \ \ (a = (b, \varepsilon) \in \mathbb{B}, z \in \overline{\mathbb{D}}).$$

For every m $(m \geq 2, m \in \mathbb{N})$ the collection $(F_a)_{a\in\mathbb{B}}$ induces a unitary representation of the Blaschke group on the space $\mathcal{B}^m(\mathbb{D})$. Namely, let us define

$$U_a^{[m]} := F_{a^{-1}}^m\, f \circ B_a^{-1} \ \ (a \in \mathbb{B}, m \in \mathbb{N}, m \geq 2, f \in \mathcal{B}^m(\mathbb{D}).$$

Then the following claim can be proved (see [6], [7]):

VI. *For all $m \in \mathbb{N}, m \geq 2$ $(U_a^{[m]})_{a\in\mathbb{B}}$ is a unitary, irreducible representation of the group $\mathbb{B}$ on the Hilbert space $\mathcal{B}^m(\mathbb{D})$.*

For the Voice-transform V_ρ the following analogue of the Plancherel formula holds.

VII. *The Voice transform induced by $(U_a)_{a\in\mathbb{B}}$ satisfies*

$$[V_{\rho_1} f, V_{\rho_2} g] = \frac{1}{2}\langle \rho_1, \rho_2\rangle\langle f, g\rangle \ \ (\rho_1, \rho_2, f, g \in \mathcal{B}^2(\mathbb{D})),$$

where

$$[F, G] := \int_{\mathbb{B}} F(a)\overline{G(a)}\, dm(a),$$

and m is the (7.13) Haar-measure of the group $\mathbb{G}$. Furthermore every $\rho \in \mathcal{H}^2(\mathbb{D})$ is admissible.

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Chapter 8

APPLICATIONS OF SIDON TYPE INEQUALITIES

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Abstract The aim of this paper is to show several areas, such as integrability conditions, strong summation, theory of multipliers, where the so called Sidon type inequalities can be applied. We will focus our attention mainly to the methods that link these areas and the Sidon type inequalities, and present some characteristic results. The bibliography at the end of the paper is far from complete but provides enough information for those who would like to learn more about these fields. The model used for the interpretation is the Walsh case but we will make remarks concerning the trigonometric case as well.

1. Introduction

Throughout this paper w_n will denote the nth Walsh function in Paley's ordering. $\widehat{f}(n)$, $S_n f$, and Sf $(f \in L^1)$ will stand for the nth Walsh-Fourier coefficient, the nth Walsh-Fourier partial sum, and Walsh-Fourier series respectively. The Walsh-Dirichlet kernels are defined as $D_n = \sum_{k=0}^{n-1} w_k$. Concerning the theory of dyadic analysis the book of Schipp, Wade and Simon [SWS90] is recommended as a general refence.

Hardy spaces will play an important role in the paper. Especially the real non periodic Hardy space $\mathcal{H}$, and the dyadic Hardy space $\mathbb{H}$. Basic results results concerning Hardy spaces can be found in the book of Kashin and Saakian [KS89]. For details on the theory of dyadic Hardy spaces and their applications we refer the reader to the monography of Weisz [Wei94]

Another type of Banach spaces that will be used in the paper are Orlicz spaces. If M is a so called Young function then L_M will denote the corresponding Or-

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licz space. The monographies of Krasnosel'skiĭ and Rutickiĭ [KR61], and of Rao and Zen [RZ91] are recommended for the theory and application of Orlicz spaces.

2. Sidon Type Inequalities

Let $c_k \in \mathbb{R}$ $(k = 1, \dots, 2^n,\ n \in \mathbb{N})$. Then the step function $\Gamma(c_1, \dots, c_{2^n})$ is defined as follows

$$\Gamma(c_1, \dots, c_{2^n})(x) = c_k \qquad \Big(\frac{k-1}{2^n} \le x < \frac{k}{2^n},\ k = 1, \dots, 2^n\Big).$$

Suppose that X is a norm in the space of dyadic step functions on $[0, 1)$.
An inequality

$$\frac{1}{2^n}\Big\|\sum_{k=1}^{2^n} c_k D_k\Big\|_1 \le C_X \|\Gamma(c_1, \dots, c_{2^n})\|_X \qquad (c_k \in \mathbb{R},\ n \in \mathbb{N}) \tag{8.1}$$

is called a Sidon type inequality.
Such an inequality was first proved by Telyakovskiĭ [Tel73] for trigonometric Dirichlet kernels with $X = L^\infty$, i.e. with $\|\Gamma(c_1, \dots, c_{2^n})\|_X = \max_{1\le k\le 2^n} |c_k|$. His result was improved by Bojanić and Stanojević [BS82] by showing that (8.1) holds for $X = L^p$ $(1 < p \le \infty)$. Another improvement, where $X = \log \alpha L_1 + \alpha^{-1/q} L_p$ $(\alpha \ge 1,\ \frac{1}{p} + \frac{1}{q} = 1)$, is due to Tanović-Miller [Tan90]. The best result along this line was given by Schipp [Sch92]. Namely, he showed that the real non periodic Hardy space $\mathcal{H}$ can stand for X in (8.1).
Móricz and Schipp [MS90] proved the Walsh analogue of the result of Bojanić and Stanojević. Then Schipp [Sch92] proved that (8.1) holds for the Walsh-Dirichlet kernels if X is the dyadic Hardy space $\mathbb{H}$. We note that Schipp considered Sidon type inequalities and Hardy norms in a more general setting, and the Walsh and the trigonometric systems are special cases only. Later the author showed [Fri00] that even in the Walsh case X can be the real non periodic Hardy space $\mathcal{H}$.
At this point we call the attention that unlike the other norms applied in Sidon type inequalities the Hardy norm is no rearrangement invariant, and there is no simple direct expression of c_k's that provides the Hardy norm of the step function $\Gamma(c_1, \dots, c_{2^n})$. Another motivation for further investigations was to find out how far is the Hardy norm variant from the best possible result. In connection with these the author [Fri95] proved that

$$\begin{aligned}\max_{p\in P_{2^n}} \|\sum_{k=1}^{2^n} c_{p_k} D_k\|_{L_1} &\approx \|\Gamma(c_1, \dots, c_{2^n})\|_{L_M} \\ &\approx \sum_{k=1}^{2^n} |c_k|\Big(1 + \log^+ \frac{|c_k|}{2^{-n}\sum_{j=1}^{2^n} |c_j|}\Big),\end{aligned} \tag{8.2}$$

where P_{2^n} is the set of permutations of $\{1,\ldots,2^n\}$, and L_M is the Orlicz space induced by the Young function $M(x) \approx x \log x$.
We note that similar result holds for the trigonometric system ([Fri93]).
This means that (8.2), in which the right side is an Orlicz norm, is the best rearrangement invariant Sidon type inequality. Also the connection with the Hardy norms and this Orlicz norm was given in [Fri93]. Namely, it was shown that the largest rearrangement invariant space included in $\mathcal{H}$ is the Orlicz space L_M. More precisely, if

$$H^\sharp = \left\{ f \in \mathcal{H} \;:\; f \circ \nu \in \mathcal{H},\; \nu \in L_0 \right\},$$

where L_0 is the set of one-to-one measure preserving maps on $[0,1)$ then

$$H^\sharp = L_M \,.$$

Moreover

$$\|f\|_{H^\sharp} = \sup\left\{ \|f\circ\nu\|_{\mathcal{H}} \;:\; \nu \in L_0 \right\} \approx \|f\|_{L_M} \approx \int_0^1 |f| \Big(1 + \log^+ \frac{|f|}{\|f\|_1} \Big) \,.$$

There have several generalizations and variants been given of the Sidon type inequalities above. For example the shifted variant of (8.2), i.e. the one that has $\sum_{k=K}^N c_k D_k$ $(K, N \in \mathbb{N})$ on the left side is in [Fri95]. Another version that has found applications is a truncated Sidon type inequality proved by Móricz [Mór90]. Its Walsh version, which was shown by Daly and the author [DF03], reads as follows

$$\int_{2^{-N}}^1 \Big| \sum_{k=1}^{2^n} c_k D_k(x) \Big| \, dx \le C 2^{N(1-1/q)} \Big(\sum_{k=1}^{2^n} |c_k|^q \Big)^{1/q}, \tag{8.3}$$

where $(n, N \in \mathbb{N},\ 1 < q \le 2)$.

As a closing remark to this section we note that however the development of Sidon type inequalities went parallel with respect to the trigonometric and the Walsh systems the techniques used for the two cases are different at several points.

3. Integrability Classes

We will consider the integrability and L^1-convergence of the Walsh series

$$\sum_{n=0}^{\infty} a_n w_n \,, \tag{8.4}$$

where (a_n) is a sequence of real numbers.
Let $\widehat{L}_W$ denote the space of real sequences for which (8.4) represents a Walsh-Fourier series of an integrable function. There is no characterization known for

$\widehat{L}_W$ in terms of the coefficients in (8.4). There are several known examples for conditions with respect to the coefficients that imply the integrability of (8.4). A subset of $\widehat{L}_W$ generated by such a condition is called integrability class.
The existence of an integrable function whose Walsh-Fourier series is (8.4) does not mean that the series converges to the function in L^1 norm. An integrability class is called an integrability and convergence class if for each element (a_n) of it the corresponding series converges in L^1-norm if and only if $\lim_{n\to\infty} |a_n| \, \|D_n\|_1 = 0$.

The connection between Sidon type inequalities and integrability and L^1-convergence classes is quite obvious. Indeed, a simple summation by parts of (8.4) leads to the series

$$\sum_{k=1}^{\infty} \Delta a_k D_k \, .$$

After breaking it into dyadic blocks a proper condition for $\| \sum_{k=2^n}^{2^{n+1}-1} \Delta a_k D_k \|_1$ would yield the desired convergence. For instance the well known integrability and convergence class due to Fomin [Fom73] in the trigonometric case, i.e. the one given by the condition

$$\sum_{n=0}^{\infty} 2^{n(1-1/p)} \Big(\sum_{k=2^n}^{2^{n+1}-1} |\Delta a_k|^p \Big)^{1/p} \leq C \quad (p > 1) \, .$$

can be deduced from the Sidon type inequalities of Bojanić and Stanojević [BS82] in the trigonometric and of Schipp and Móricz [MS90] in the Walsh case.
Let us mention some other classical integrability and convergence conditions. Young [You13] showed that the set of convex null sequences forms an integrability and convergence class. The same holds for the set of so called quasi convex sequences as was shown by Komogorov [Kol23]. Another condition was given by Sidon [Sid39]. His result was reformulated by Telyakovskiĭ [Tel73]. This class contains those null sequences (a_k) for which there exists a non increasing sequence (A_k) such that $\sum_{k=0}^{\infty} A_k < \infty$, and $|\Delta a_k| \leq A_k$. It was shown by he author [Fri96] that also these classical results can be deduced from proper known Sidon type inequalities.
We note that there exist integrability and convergence conditions that can not be originated from a Sidon type inequality. In connection with such results we refer the reader to the papers of Aubertin and Fournier [AF93], [AF94] and of Buntinas and Tanović-Miller [BT90]. There is yet another well-known condi-

tion which turned out to be connected with a Sidon type inequality. Namely,

$$\sum_{n=2}^{\infty}\Big|\sum_{k=1}^{[n/2]}\frac{\Delta a_{n-k}-\Delta a_{n+k}}{k}\Big|<\infty \quad , \quad \sum_{n=1}^{\infty}|\Delta a_n|<\infty\,. \tag{8.5}$$

is an integrability and convergence condition due to Telyakovskiĭ [Tel64]. We remark that it is an improvement of an earlier condition of Boas [Boa56], and was proved for cosine series originally.
The author [Fri01] proved that the formula in (8.5) can be understood as a sequence norm. More precisely, it is equivalent to a sequence Hardy norm which has an atomic structure, and the atoms there are related to those of $\mathcal{H}$. This interpretation of (8.5) made possible the extension of the Telyakovskiĭ condition to other cases including Walsh series.
Let us now suppose that $\sum_{n=0}^{\infty} a_n w_n$ is a Walsh-Fourier series, i.e. $a_k = \widehat{f}_k$ with some $f \in L^1$, and consider its L^1 convergence. Comparing it with the previous setting here the integrability is already guaranteed and only the L^1 convergence is the question. A major difference is that in this case the Fejér means of the Walsh-Fourier series are convergent. So are the generalized de la Vallée Poussin means

$$V_{n,\lambda}f=\frac{1}{[\lambda n]-n+1}\sum_{k=n}^{[\lambda n]}S_kf \qquad (n\in\mathbb{P},\,\lambda>1,\,f\in L_1). \tag{8.6}$$

Then it is enough to provide condition for the convergence of $V_{n,\lambda}f - S_nf$. By manipulating this difference the left side of the Sidon inqualities (8.1) will come up. This is how the L^1 convergence problem is linked to Sidon type inequalities. Typical results in this line are the so called Hardy-Karamata type Tauberian conditions. For instance Bojanić and Stanojević [BS82] showed that if

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}k^{p-1}|\Delta\widehat{f}(k)|^p=0 \qquad (p>1)$$

then $\lim_{n\to\infty}\|f-S_nf\|_1=0$ if and only if $\lim_{n\to\infty}\widehat{f}_n\log n=0$. The Walsh version was proved by Móricz [Mór89].
Another condition, namely

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}|\Delta\widehat{f}(k)|\log k=0\,.$$

was given by Grow and Stanojević [GS95].
Then the author [Fri97] gave the following condition, which subsumes the pre-

vious ones

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}|\Delta\widehat{f}(k)|\log^+|k\Delta\widehat{f}(k)|=0\,. \tag{8.7}$$

If (8.7) holds then $\lim_{n\to\infty}\|f-S_nf\|_{L_1}=0$ if and only if $\lim_{n\to\infty}\widehat{f}(n)L_n=0$, where L_n stands for the nth Walsh-Lebesgue constant. Moreover if $f\in\mathbb{H}$ then $\lim_{n\to\infty}\|f-S_nf\|_{\mathbb{H}}=0$ if and only if $\lim_{n\to\infty}\widehat{f}(n)L_n=0$. We note that the trigonometric version of (8.7) can be found in [Fri97'].

4. Strong Summation

Let us start this section with the classical result of Hardy and Littlewood [HL13] on strong summability of trigonometric Fourier series of continuous functions:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|S_kf(x)-f(x)|^q=0 \qquad (q>1). \tag{8.8}$$

The connection between strong summability of continuous functions and Sidon type inequalities relies on a duality relation. Suppose that X is a homogeneous Banach spaces that contains the set of dyadic step functions. Let Y denote the dual of X.

The Y strong means of a continuous function f at x is defined as

$$\big\|\Gamma\big(S_1f(x),\dots,S_{2^n}f(x)\big)\big\|_Y.$$

For instance the strong mean in (8.8) corresponds to $Y=L^q$.

The following duality result is proved in [FS95], and [FS98] in a more general setting:

$$\begin{gathered}\sup_x\big\|\Gamma\big(S_1f(x),\dots,S_{2^n}f(x)\big)\big\|_Y\le C_X\|f\|_{L_\infty}\\ \text{if and only if}\\ \Big\|\frac{1}{2^n}\sum_{k=1}^{2^n}c_kD_k\Big\|_{L_1}\le C_X\|\Gamma(c_1,\dots,c_{2^n})\|_X\,.\end{gathered} \tag{8.9}$$

Here f is a continuous function in the trigonometric case, while dyadically continuous function in the Walsh case, and the c_k-s are arbitrary real numbers. For example the trigonometric Sidon type inequality of Bojanic and Stanojević [BS82] is dual to the strong summability result of Hardy and Littlewood [HL13]. Concerning strong summation and approximation by trigonometric Fourier series we refer the reader to the monograph of Leindler [Lei85] as a general reference.

If L_M is the Orlicz space induced by the Young function $M(x)\approx x\log x$ then its dual is L_N where $N = e^x - 1$. On this bases we deduced [FS98] from the Sidon inequality in (8.2) that for any $A > 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(e^{A|S_k f(x) - f(x)|} - 1 \right) = 0 \tag{8.10}$$

in both the trigonometric and the Walsh cases. Moreover, utilizing the fact that (8.2) is the best rearrangement invariant Sidon type inequality for the trigonometric and the Walsh systems we could prove [FS98] the sharpness of (8.10) in the following sense: If $\varphi \nearrow$, $\varphi(0) = 0$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(|S_k f(x) - f(x)|) = 0 \tag{8.11}$$

then there exists $A > 0$ such that $\varphi(x) \leq e^{Ax}$. We note that (8.10) and its sharpness was originally proved by Totik [Tot80] for continuous φ in (8.11).
The rest of this section is on strong approximation properties of Walsh, and trigonometric Fourier series. First we note that the investigation of strong approximation properties of trigonometric Fourier series was started by Alexits and Králik [AK63]. Let us start with modifying the notation of de la Vallée Poussin means given in (8.6) as follows

$$V_{r,n} f = \frac{1}{r} \sum_{k=1}^{r} S_{k+n} f \qquad (r, n \in \mathbb{N}).$$

Let Y be as before. Then we can define the concept of strong Y oscillations of Fourier partial sums as follows

$$\left\| \Gamma\big(S_{1+n} f(x) - V_{1,n} f(x), \ldots, S_{r+n} f(x) - V_{r,n} f(x)\big) \right\|_Y \qquad (r, n \in \mathbb{N}).$$

The error of best approximation of f by Walsh polynomials of order at most n is defined as follows

$$E_n f = \inf_{p \in \mathcal{P}_n} \|f - p\|_\infty \qquad (n \in \mathbb{P}),$$

where $\mathcal{P}_n$ is the set of Walsh polynomials of order not greater than n.
Schipp and the author [FS98] showed that that the following duality relation

holds between strong oscillation and shifted Sidon type inequalities.

$$\frac{1}{2^n}\|\sum_{k=1}^{2^n} c_k D_{k+j2^n}\|_1 \le C_X \|\Gamma(c_1,\dots,c_{2^n})\|_X$$
$$\text{with} \quad \sum_{k=1}^{2^n} c_k = 0 \quad (j,n\in\mathbb{N})$$

if and only if

$$\big\|\Gamma\big(S_{1+j2^n}f(x)-V_{2^n,j2^n}f(x),\dots,S_{(j+1)2^n}f(x)-V_{2^n,j2^n}f(x)\big)\big\|_Y \le C_Y E_{j2^n} f \quad (j,n\in\mathbb{N})\,. \tag{8.12}$$

On the basis of this duality relation all we need is to estimate $|f(x)-V_{2^n,j2^n}f(x)|$ in order that we can deduce estimations for the generalized strong means, i.e. for

$$\big\|\Gamma\big(S_{j2^n+1}f(x)-f(x),\dots,S_{(j+1)2^n}f(x)-f(x)\big)\big\|_Y \qquad (n\in\mathbb{N})\,.$$

In other words the only additional information needed is the rate of convergence of generalized de la Vallée Poussin means.
There are several results concerning strong approximation that can be deduced from (8.12). Here we mention only one and refer the reader for further results to [FS98].
Let $\varphi:[0,\infty)\mapsto\mathbb{R}$ be a monotonically increasing continuous function with $\lim_{u\to 0^+}\varphi(u)=0$ for which there exists A such that

$$\varphi(u)\le \exp(Au) \qquad (u\ge 0)\,,$$

and

$$\varphi(2u)\le A\varphi(u) \qquad (0<u<1).$$

Then

$$\frac{1}{n}\sum_{k=n+1}^{2n}\varphi\big(\big|S_kf(x)-f(x)\big|\big)\le C\varphi(E_nf) \qquad (n\in\mathbb{P}).$$

The trigonometric version of this estimate was given by Totik in [Tot85].

5. Multipliers

Let φ be a sequence of real numbers. Then the multiplier operator T_ϕ is defined as

$$T_\phi f=\sum_{k=0}^{\infty}\phi(k)\widehat{f}(k)w_k \qquad (f\in L^1).$$

The operator T_ϕ can be considered as a convolution operator with generalized kernel $\sum_{k=0}^{\infty}\phi(k)w_k$. It is known that the operator norm from L^1 to itself can be estimated by the L^1 norm of the kernel. This is how integrability conditions, and therefore Sidon type inequalities are connected with L^1 multiplier conditions. For instance the Telyakovskiĭ type integrability condition proved in [Fri01] implies the multiplier condition [DF03]:
If

$$\sum_{n=1}^{\infty}|\Delta\phi(n)| + \sum_{n=2}^{\infty}\Big|\sum_{k=1}^{[n/2]}\frac{\Delta\phi(n-k)-\Delta\phi(k+n)}{k}\Big| < \infty$$

then T_ϕ is bounded on L^1 and on $\mathbb{H}$.
In connection with this we note that the well known Marcinkiewicz condition [Mar39], i.e.

$$\phi \text{ bounded, and } \sum_{k=2^n}^{2^{n+1}-1}|\Delta\phi(k)| < C$$

is sufficient for the boundedness of T_ϕ on L^p $(1<p<\infty)$ but is not for L^1 in the trigonometric case. Wo-Sang Young [WSY94] showed this result extends to the setting of Vilenkin groups; and thus, in particular valid in the Walsh case. Daly and the author [DF03] showed that even the stronger condition that ϕ is of bounded variation does not imply that T_ϕ is bonded on L^1 or on $\mathbb{H}$.
Let us turn to the problem of boundedness of multiplier operators on $\mathbb{H}^p$ $(0<p<1)$. We will consider the classical coefficient condition, the so called Hörmander-Mihlin condition [Hör60] and [Mih56]. It is known to be sufficient in the spaces L^p $(1<p<\infty)$ even in the multidimensional case. For details we refer the reader to the monograph of Edwards and Gaudry [EG77]. The condition reads as follows

$$2^j\Big(\sum_{k=2^j+1}^{2^{j+1}}\frac{|\Delta\phi(k)|^r}{2^j}\Big)^{1/r} \le C \qquad (j\in\mathbb{N}). \tag{8.13}$$

We note that a multiplier that satisfies the Hörmander-Mihlin condition with any $r\ge 1$ satisfies also the Marcinkiewicz condition and that $L^p=\mathbb{H}^p$ for $p>1$.
Daly and the author [DF03] showed that unlike the Marcinkiewicz condition the the Hörmander-Mihlin condition extends to Hardy spaces $\mathbb{H}^p$ for $p<1$. Namely we proved that if ϕ is bounded and satisfies (8.13) with $1<r\le 2$, and $p>r/(2r-1)$ then T_ϕ is bounded from $\mathbb{H}^p$ to itself. This in particular means that for any $1<r\le 2$ (8.13) is sufficient for the boundedness of T_ϕ on $\mathbb{H}^1$. On the negative side we showed that for any $1\le r\le\infty$ there exists a bounded multiplier φ that satisfies (8.13) with the natural modification for $r=\infty$, but T_φ is not bounded on L_1. This result shows a major difference

between L^1 and $\mathbb{H}^1$.
We also proved in [DF03] hat our result is sharp in the sense that if $p < r/(2r-1)$ then there exists a bounded multiplier ϕ that satisfies (8.13) but T_φ is not bounded from $\mathbb{H}^p$ to L_p. For the trigonometric version of these results see the paper of Daly and the author [DF05].
We note that existing multiplier theorems for Hardy spaces give growth conditions on the dyadic blocks of the Walsh series of the kernel, see e.g. Daly and Phillips [DPh98], [DPh98'], Kitada [Kit87], Onneweer and Quek [OQ89], and Simon [Sim85], but these growth are not computable directly in terms of the coefficients.

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Chapter 9

SUMMABILITY OF WALSH-FOURIER SERIES AND THE DYADIC DERIVATIVE

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Abstract In this survey paper we present some results on summability of one- and multi-dimensional Walsh-Fourier series and on the dyadic derivative. Three summability methods, the Fejér, Cesàro and Riesz methods are investigated. In the multi-dimensional case three types of convergence are considered, the restricted, the unrestricted and the Marcinkiewicz-type. We will prove that the maximal operator of the summability means is bounded from the martingale Hardy space H_p to L_p $(p > p_0)$. For $p = 1$ we obtain a weak type inequality by interpolation, which ensures the a.e. convergence of the summability means. Similar results are formulated for the one- and multi-dimensional dyadic or Gibbs derivative. The dyadic version of the classical theorem of Lebesgue is proved, more exactly, the dyadic derivative of the dyadic integral of a function f is a.e. f.

1. Introduction

In this paper we will consider summation methods for one-dimensional and multi-dimensional Walsh-Fourier series and the one- and multi-dimensional dyadic derivative. First we present the corresponding results for trigonometric Fourier series and then the extensions to Walsh-Fourier series. Three types of summability methods will be investigated, the Fejér, Cesàro or (C, α) and the Riesz methods. The Fejér summation is a special case of the Cesàro method, $(C, 1)$ is exactly the Fejér method. In the multi-dimensional case three types of convergence and maximal operators are considered, the restricted (convergence over the diagonal or over a cone), the unrestricted (convergence over

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$\mathbf{N}^d$) and the Marcinkiewicz-type. We introduce martingale Hardy spaces H_p and prove that the maximal operators of the summability means are bounded from H_p to L_p whenever $p > p_0$ for some $p_0 < 1$. For $p = 1$ we obtain a weak type inequality by interpolation, which implies the a.e. convergence of the summability means. The a.e. convergence and the weak type inequality are proved usually with the help of a Calderon-Zygmund type decomposition lemma. However, this lemma does not work in higher dimensions. Our method, that can be applied in higher dimension, too, can be regarded as a new method to prove the a.e. convergence and weak type inequalities.

Similar results are formulated for the one- and multi-dimensional dyadic or Gibbs derivative. We get that the maximal operators are bounded from H_p to L_p if $p > p_0$ ($p_0 < 1$) and a weak type inequality if $p = 1$. This implies the dyadic version of the classical theorem of Lebesgue, more exactly, the dyadic derivative of the dyadic integral of a function f is a.e. f. In this survey paper we summarize the results appeared in this topic in the last 10–20 years. The analogous results of this paper are proved for the trigonometric, Vilenkin and Ciesielski systems in Weisz [93, 91].

2. One-dimensional Walsh-Fourier Series

The well known Carleson's theorem [8] says, that the partial sums $s_n f$ of the trigonometric Fourier series of a one-dimensional function $f \in L_2(\mathbf{T})$ converge a.e. to f as $n \to \infty$. Later Hunt [35] extended this result to all $1 < p < \infty$. This theorem does not hold, if $p = 1$. However, if we take some summability methods, we can obtain convergence for L_1 functions, too.

In 1904 Fejér [15] investigated the arithmetic means of the partial sums, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $f(x)$. One year later Lebesgue [38] extended this theorem and obtained that every integrable function is Fejér summable at each Lebesgue point, thus a.e. The Cesàro means or (C, α) $(\alpha > 0)$ means are generalizations of the Fejér means; if $\alpha = 1$ then the two types of means are the same. M. Riesz [50] proved that the (C, α) $(\alpha > 0)$ means $\sigma_n^\alpha f$ of a function $f \in L_1(\mathbf{T})$ converge a.e. to f as $n \to \infty$ (see also Zygmund [98, Vol. I, p.94]). Moreover, it is known that the maximal operator of the (C, α) means $\sigma_*^\alpha := \sup_{n \in \mathbf{N}} |\sigma_n^\alpha|$ is of weak type $(1, 1)$, i.e.,

$$\sup_{\rho > 0} \rho \lambda(\sigma_* f > \rho) \le C \|f\|_1 \qquad (f \in L_1(\mathbf{T})).$$

This result can be found implicitly in Zygmund [98, Vol. I, pp. 154-156].

For the Fejér means Móricz [43] and Weisz [78] verified that σ_*^1 is bounded from $H_1(\mathbf{T})$ to $L_1(\mathbf{T})$. The author [82] extended this result to the Cesàro summation, i.e. to σ_*^α, $\alpha > 0$ and $1/(\alpha+1) < p < \infty$.

Stein and Weiss [66] and Butzer and Nessel [5] proved for $\gamma = 1, 2$ that the Riesz means $\sigma_n^{\alpha,\gamma} f$ of a function $f \in L_1(\mathbf{T})$ converge a.e. to f as $n \to \infty$. The author [83] verified the same result for all $\gamma \geq 1$ and that the maximal Riesz operator $\sigma_*^{\alpha,\gamma} := \sup_{n\in\mathbf{N}} |\sigma_n^{\alpha,\gamma}|$ is of weak type $(1,1)$. Moreover, we proved in [83] that $\sigma_*^{\alpha,\gamma}$ is bounded from $H_p(\mathbf{T})$ to $L_p(\mathbf{T})$ provided that $1/(\alpha+1) < p < \infty$ and $0 < \alpha \leq 1$. In the special case $\alpha = \gamma = 1$ the Riesz means are exactly the Fejér means.

In the next subsections analogous results will be given for Walsh-Fourier series.

Dyadic Hardy spaces

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^j$ be its Cartesian product $\mathbf{X} \times \ldots \times \mathbf{X}$ taken with itself j-times. We briefly write $L_p[0,1)^j$ instead of the space $L_p([0,1)^j, \lambda)$ $(j \geq 1)$ where λ is the Lebesgue measure.

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbf{N}$, $0 \leq k < 2^n$. Given $n \in \mathbf{N}$ and $x \in [0,1)$ let $I_n(x)$ be the dyadic interval of length 2^{-n} which contains x. The σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in [0,1)\}$ will be denoted by $\mathcal{F}_n$ $(n \in \mathbf{N})$.

We investigate the class of *martingales* $f = (f_n, n \in \mathbf{N})$ with respect to $(\mathcal{F}_n, n \in \mathbf{N})$. The *maximal function* of a martingale f is defined by

$$f^* := \sup_{n\in\mathbf{N}} |f_n|.$$

For $0 < p \leq \infty$ the *martingale Hardy space* $H_p[0,1)$ consists of all one-parameter martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

Recall that the Hardy and L_p spaces are equivalent, if $p > 1$, in other words,

$$H_p[0,1) \sim L_p[0,1) \qquad (1 < p \leq \infty).$$

Moreover, the martingale maximal function is of weak type $(1,1)$:

$$\|f\|_{H_{1,\infty}} := \sup_{\rho>0} \rho\lambda(f^* > \rho) \leq C\|f\|_1 \qquad (f \in L_1[0,1))$$

(see Neveu [45] or Weisz [73]) and $H_1[0,1) \subset L_1[0,1)$.

Now some boundedness theorems for Hardy spaces are given. To this end we introduce the definition of the atoms. The *atomic decomposition* is a useful characterization of the Hardy spaces by the help of which some boundedness results, duality theorems, maximal inequalities and interpolation results can be proved. The atoms are relatively simple and easy to handle functions. If we have an atomic decomposition, then we have to prove several theorems for atoms, only. A first version of the atomic decomposition was introduced by Coifman and Weiss [11] in the classical case and by Herz [34] in the martingale case.

A function $a \in L_\infty$ is called a *p-atom* if

(a) $\operatorname{supp} a \subset I$, $I \subset [0,1)$ is a dyadic interval,

(b) $\|a\|_\infty \leq |I|^{-1/p}$,

(c) $\int_I a(x)\,dx = 0$.

The basic result of atomic decomposition is the following one.

THEOREM 9.1 *A martingale f is in $H_p[0,1)$ $(0 < p \leq 1\})$ if and only if there exist a sequence $(a^k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\begin{aligned} &\sum_{k=0}^{\infty} \mu_k a^k = f \quad \textit{in the sense of martingales,} \\ &\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \end{aligned} \tag{9.1}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p} \tag{9.2}$$

where the infimum is taken over all decompositions of f of the form (9.1).

The proof of this theorem can be found e.g. in Latter [37], Lu [39], Coifman and Weiss [11], Coifman [10], Wilson [94, 95] and Stein [65] in the classical case and in Weisz [73] for martingale Hardy spaces.

If I is a dyadic interval then let $I^r = 2^r I$ be a dyadic interval, for which $I \subset I^r$ and $|I^r| = 2^r |I|$ $(r \in \mathbf{N})$.

The following result gives a sufficient condition for V to be bounded from $H_p[0,1)$ to $L_p[0,1)$. For $p_0 = 1$ it can be found in Schipp, Wade, Simon and Pál [56] and in Móricz, Schipp and Wade [44], for $p_0 < 1$ see Weisz [78].

THEOREM 9.2 *Suppose that*

$$\int_{[0,1)\setminus I^r} |Va|^{p_0}\, d\lambda \leq C_{p_0}$$

for all p_0-atoms a and for some fixed $r \in \mathbf{N}$ and $0 < p_0 \leq 1$. If the sublinear operator V is bounded from $L_{p_1}[0,1)$ to $L_{p_1}[0,1)$ $(1 < p_1 \leq \infty)$ then

$$\|Vf\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)) \tag{9.3}$$

for all $p_0 \le p \le p_1$. *Moreover, if* $p_0 < 1$ *then the operator* V *is of weak type* $(1,1)$, *i.e. if* $f \in L_1[0,1)$ *then*

$$\sup_{\rho>0} \rho\lambda(|Vf| > \rho) \le C\|f\|_1. \tag{9.4}$$

Note that (9.4) can be obtained from (9.3) by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [2] and Bennett and Sharpley [1] or Weisz [73, 91]. The interpolation of martingale Hardy spaces was worked out in [73]. Theorem 9.2 can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

We formulate also a weak version of this theorem.

THEOREM 9.3 *Suppose that*

$$\sup_{\rho>0} \rho^p \, \lambda\Big(\{|Va| > \rho\} \cap \{[0,1) \setminus I^r\}\Big) \le C_p$$

for all p*-atoms* a *and for some fixed* $r \in \mathbf{N}$ *and* $0 < p < 1$. *If the sublinear operator* V *is bounded from* L_{p_1} *to* L_{p_1} $(1 < p_1 \le \infty)$, *then*

$$\|Vf\|_{p,\infty} \le C_p \|f\|_{H_p} \quad (f \in H_p[0,1)).$$

Walsh functions

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbf{N}).$$

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*:

$$w_n := \prod_{k=0}^{\infty} {r_k}^{n_k},$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad (0 \le n_k < 2).$$

If $f \in L_1[0,1)$ then the number

$$\hat{f}(n) := \int_{[0,1)} f w_n \, d\lambda \qquad (n \in \mathbf{N})$$

is said to be the nth *Walsh-Fourier coefficient* of f. We can extend this definition to martingales as well in the usual way (see Weisz [74]). Denote by $s_n f$ the nth *partial sum* of the Walsh-Fourier series of a martingale f, namely,

$$s_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k.$$

It is known that $s_{2^n} f = f_n \; (n \in \mathbf{N})$ and

$$s_{2^n} f \to f \qquad \text{in } L_p\text{-norm and a.e. as } n \to \infty,$$

if $f \in L_p[0,1) \; (1 \leq p < \infty)$.

The Carleson's theorem was extended to Walsh-Fourier series by Billard [3] and Sjölin [64]:

$$s_n f \to f \qquad \text{a.e. as } n \to \infty, \tag{9.5}$$

whenever $f \in L_p[0,1) \; (1 < p < \infty)$. If

$$s_* f := \sup_{n \in \mathbf{N}} |s_n f|$$

denotes the *maximal partial sum operator*, then

$$\|s_* f\|_p \leq C_p \|f\|_p \qquad (f \in L_p[0,1), 1 < p < \infty). \tag{9.6}$$

This implies besides the a.e. convergence (9.5) also the L_p-norm convergence of $s_n f$ to f $(1 < p < \infty)$. These theorems do not hold, if $p = 1$, however they can be generalized for $p = 1$ with the help of some summability methods. Fine [16] proved that the Cesàro or (C, α) means $\sigma_n^\alpha f$ $(\alpha > 0)$ of a function $f \in L_1[0,1)$ converge a.e. to f as $n \to \infty$. The convergence at all Walsh-Lebesgue points was verified by the author [72]. It is known that the maximal operator of the Fejér means $(\alpha = 1)$ is of weak type $(1,1)$, (see Schipp [52]). Fujii [19] proved that σ_*^1 is bounded from $H_1[0,1)$ to $L_1[0,1)$ (see also Schipp, Simon [54]). For Vilenkin-Fourier series these results are due to Simon [58]. For the Riesz means of Walsh-Fourier series it was known only that the one-dimensional maximal Riesz operator is bounded from $L_p[0,1)$ to $L_p[0,1)$ $(1 < p < \infty)$ (see Paley [49]).

Summability of one-dimensional Walsh-Fourier series

The *Fejér*, *Cesàro* (or (C, α)) and *Riesz means* of a martingale f are given by

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} s_k f = \sum_{k=0}^{n-1} \Big(1 - \frac{k}{n}\Big) \hat{f}(k) w_k,$$

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_k f = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha \hat{f}(k) w_k$$

and

$$\sigma_n^{\alpha,\gamma} f := \frac{1}{n^{\alpha\gamma}} \sum_{k=0}^{n-1} \Big((n^\gamma - k^\gamma)^\alpha \Big) \hat{f}(k) w_k,$$

respectively, where

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!}.$$

If $\alpha = 1$ then $\sigma_n^\alpha f = \sigma_n f$, and so the $(C,1)$ means are the Fejér means. Since $A_k^\alpha \sim k^\alpha$ $(k \in \mathbf{N})$, we have $A_{n-k-1}^\alpha \sim (n-k)^\alpha$. Thus the means $\sigma_n^\alpha f$ are "similar" to the ones $\sigma_n^{\alpha,1} f$.

The *maximal operator* of the Cesàro and Riesz means are defined by

$$\sigma_*^\alpha f := \sup_{n \in \mathbf{N}} |\sigma_n^\alpha f|, \qquad \sigma_*^{\alpha,\gamma} f := \sup_{n \in \mathbf{N}} |\sigma_n^{\alpha,\gamma} f|.$$

In what follows we use a common notation σ_n for the Cesàro and Riesz means and σ_* for the corresponding maximal operators.

The next result generalizes (9.6) for the maximal operator of the summability means (see Zygmund [98] and Paley [49]).

THEOREM 9.4 *If* $0 < \alpha \le 1 \le \gamma$ *and* $1 < p \le \infty$ *then*

$$\|\sigma_* f\|_p \le C_p \|f\|_p \qquad (f \in L_p[0,1)).$$

Moreover, for all $f \in L_p[0,1)$ $(1 < p < \infty)$,

$$\sigma_n f \to f \qquad \text{a.e. and in } L_p\text{-norm as } n \to \infty.$$

The L_p-norm convergence holds also, if $p = 1$. Applying Theorems 9.2 and 9.3, we extended the previous result to $p < 1$ in [74, 89, 91, 63] (for $\alpha = p = 1$ see Fujii [19]):

THEOREM 9.5 *If* $0 < \alpha \le 1 \le \gamma$ *and* $1/(\alpha+1) < p \le \infty$ *then*

$$\|\sigma_* f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p[0,1))$$

and for $f \in H_{1/(\alpha+1)}[0,1)$,

$$\|\sigma_* f\|_{1/(\alpha+1),\infty} = \sup_{\rho > 0} \rho \lambda(\sigma_* f > \rho)^{\alpha+1} \le C \|f\|_{H_{1/(\alpha+1)}}.$$

The critical index is $p = 1/(\alpha+1)$, if p is smaller than or equal to this critical index, then σ_* is not bounded anymore (see Simon and Weisz [63] and Simon [59]):

THEOREM 9.6 *The operator* σ_* $(0 < \alpha \leq 1 \leq \gamma)$ *is not bounded from* $H_p[0,1)$ *to* $L_p[0,1)$ *if* $0 < p \leq 1/(\alpha+1)$.

We get the next weak type $(1,1)$ inequality from Theorem 9.5 by interpolation (Weisz [74, 89, 91], for $\alpha = 1$ Schipp [52]).

COROLLARY 9.1 *If* $0 < \alpha \leq 1 \leq \gamma$ *and* $f \in L_1[0,1)$ *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_* f > \rho) \leq C\|f\|_1.$$

Since the set of the Walsh polynomials is dense in $L_1[0,1)$, Corollary 9.1 and the usual density argument (see Marcinkievicz, Zygmund [41]) imply

COROLLARY 9.2 *If* $0 < \alpha \leq 1 \leq \gamma$ *and* $f \in L_1[0,1)$ *then*

$$\sigma_n f \to f \qquad \text{a.e. as } n \to \infty.$$

Recall that this convergence result was proved by Fine [16] for the (C,α) summation and in this general version by the author [74, 89]. With the help of the conjugate functions we proved also

THEOREM 9.7 *If* $0 < \alpha \leq 1 \leq \gamma$ *and* $1/(\alpha+1) < p \leq \infty$ *then*

$$\|\sigma_n f\|_{H_p} \leq C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)).$$

COROLLARY 9.3 *If* $0 < \alpha \leq 1 \leq \gamma$, $1/(\alpha+1) < p < \infty$ *and* $f \in H_p[0,1)$ *then*

$$\sigma_n f \to f \qquad \text{in } H_p\text{-norm as } n \to \infty.$$

Some of these theorems can be found in Gát [21] and Simon [60, 59] for the Walsh-Kaczmarz system, in Simon [58] and Weisz [80] for the Vilenkin systems.

Note that for $\alpha > 1$ the results can be reduced to the $\alpha = 1$ case.

3. The Dyadic Derivative

The one-dimensional differentiation theorem due to Lebesgue

$$f(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \qquad \text{a.e.} \qquad (f \in L_1[0,1))$$

is well known (see e.g. Zygmund [98]).

In this section the dyadic analogue of this result will be formulated. Gibbs [26], Butzer and Wagner [6, 7] introduced the concept of the *dyadic derivative* as follows. For each function f defined on $[0,1)$ set

$$(\mathbf{d}_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x\dot{+}2^{-j-1})), \qquad (x \in [0,1)).$$

Then f is said to be *dyadically differentiable* at $x \in [0,1)$ if $(\mathbf{d}_n f)(x)$ converges as $n \to \infty$. It was verified by Butzer and Wagner [7] that every Walsh function is dyadically differentiable and

$$\lim_{n\to\infty} (\mathbf{d}_n w_k)(x) = k w_k(x) \qquad (x \in [0,1), k \in \mathbf{N}).$$

Let W be the function whose Walsh-Fourier coefficients satisfy

$$\hat{W}(k) := \begin{cases} 1, & \text{if } k = 0 \\ 1/k, & \text{if } k \in \mathbf{N}, k \neq 0. \end{cases}$$

The *dyadic integral* of $f \in L_1[0,1)$ is introduced by

$$\mathbf{I}f(x) := f * W(x) := \int_0^1 f(t) W(x\dot{+}t)\, dt.$$

Notice that $W \in L_2[0,1) \subset L_1[0,1)$, so $\mathbf{I}$ is well defined on $L_1[0,1)$.

Let the *maximal operator* be defined by

$$\mathbf{I}_* f := \sup_{n\in\mathbf{N}} |\mathbf{d}_n(\mathbf{I}f)|.$$

The boundedness of $\mathbf{I}_*$ from $L_p[0,1)$ to $L_p[0,1)$ $(1 < p \leq \infty)$ is due to Schipp [51]:

THEOREM 9.8 *If* $1 < p < \infty$ *then*

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_p \qquad (f \in L_p[0,1)).$$

Schipp and Simon [54] verified that $\mathbf{I}_*$ is bounded from $L \log L[0,1)$ to $L_1[0,1)$. Recall that $L \log L[0,1) \subset H_1[0,1)$. These results are extended to $H_p[0,1)$ spaces in the next theorem (see Weisz [81]).

THEOREM 9.9 *Suppose that* $f \in H_p[0,1) \cap L_1[0,1)$ *and*

$$\int_0^1 f(x)\, dx = 0. \tag{9.7}$$

Then

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_{H_p}$$

for all $1/2 < p < \infty$.

We get by interpolation

COROLLARY 9.4 *If* $f \in L_1[0,1)$ *satisfies (9.7), then*

$$\sup_{\rho>0} \rho\, \lambda(\mathbf{I}_* f > \rho) \leq C\|f\|_1.$$

The dyadic analogue of the Lebesgue's differentiation theorem follows easily from the preceding weak type inequality:

COROLLARY 9.5 *If* $f \in L_1[0,1)$ *satisfies (9.7), then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \quad \textit{a.e., as } n \to \infty.$$

Corollaries 9.4 and 9.5 are due to Schipp [51] (see also Weisz [76]).

4. More-dimensional Walsh-Fourier Series

The analogue of the Carleson's theorem does not hold in higher dimensions. However, the summability results above can be generalized for the more-dimensional case. For multi-dimensional trigonometric Fourier series Zygmund [98] verified that if $f \in L(\log L)^{d-1}(\mathbf{T}^d)$ then the Cesàro means $\sigma_n^\alpha f$ converges to f a.e. and if $f \in L_p[0,1)^d$ $(1 \leq p < \infty)$ then $\sigma_n^\alpha f \to f$ in $L_p[0,1)^d$ norm as $\min(n_1,\ldots,n_d) \to \infty$. Moreover, if n must be in a cone then the a.e. convergence holds for all $f \in L_1(\mathbf{T}^d)$. More exactly, Marcinkievicz and Zygmund [41] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbf{T}^d)$ converge a.e. to f as $\min(n_1,\ldots,n_d) \to \infty$ provided that n is in a positive cone, i.e., provided that $2^{-\tau} \leq n_i/n_j \leq 2^\tau$ for every $i,j = 1,\ldots,d$ and for some $\tau \geq 0$ $(n = (n_1,\ldots,n_d) \in \mathbf{N}^d)$.

In the next subsections Walsh- and multi-dimensional analogues of the results above will be given.

d-dimensional dyadic Hardy spaces

By a *dyadic rectangle* we mean a Cartesian product of d dyadic intervals. For $n \in \mathbf{N}^d$ and $x \in [0,1)^d$ let $I_n(x) := I_{n_1}(x_1) \times \ldots \times I_{n_d}(x_d)$, where $n = (n_1,\ldots,n_d)$ and $x = (x_1,\ldots,x_d)$. The σ-algebra generated by the dyadic rectangles $\{I_n(x) : x \in [0,1)^d\}$ will be denoted again by $\mathcal{F}_n$ $(n \in \mathbf{N}^d)$.

For d-parameter martingales $f = (f_n, n \in \mathbf{N}^d)$ with respect to $(\mathcal{F}_n, n \in \mathbf{N}^d)$ we introduce three kinds of maximal functions and Hardy spaces. The *maximal functions* are defined by

$$f^\circ := \sup_{n\in\mathbf{N}} |f_{\mathbf{n}}|, \qquad f^* := \sup_{n\in\mathbf{N}^d} |f_n|,$$

where $\mathbf{n} := (n, \ldots, n) \in \mathbf{N}^d$ for $n \in \mathbf{N}$. In the first maximal function we have taken the supremum over the diagonal, in the second one over $\mathbf{N}^d$. Let E_n denote the conditional expectation operator with respect to $\mathcal{F}_n$. Obviously, if $f \in L_1[0,1)^d$ then $(E_n f, n \in \mathbf{N}^d)$ is a martingale. In the third maximal function the supremum is taken over $d-1$ indices: for fixed x_i we define

$$f^i(x) := \sup_{n_k \in \mathbf{N}, k=1,\ldots,d; k \neq i} |E_{n_1} \ldots E_{n_{i-1}} E_{n_{i+1}} \ldots E_{n_d} f(x)|.$$

For $0 < p \leq \infty$ the *martingale Hardy spaces* $H_p^\circ[0,1)^d$, $H_p[0,1)^d$ and $H_p^i[0,1)^d$ consists of all d-parameter martingales for which

$$\|f\|_{H_p^\circ} := \|f^\circ\|_p < \infty, \qquad \|f\|_{H_p} := \|f^*\|_p < \infty, \qquad \|f\|_{H_p^i} := \|f^i\|_p < \infty,$$

respectively. One can show (see Weisz [73]) that $L(\log L)^{d-1}[0,1)^d \subset H_1^i[0,1)^d \subset H_{1,\infty}[0,1)^d$ $(i = 1, \ldots, d)$, more exactly,

$$\|f\|_{H_{1,\infty}} := \sup_{\rho > 0} \rho \lambda(f^* > \rho) \leq C \|f\|_{H_1^i} \qquad (f \in H_1^i[0,1)^d)$$

and

$$\|f\|_{H_1^i} \leq C + C \||f|(\log^+ |f|)^{d-1}\|_1 \qquad (f \in L(\log L)^{d-1}[0,1)^d)$$

where $\log^+ u = 1_{\{u>1\}} \log u$. Moreover, it is known that

$$H_p^\circ[0,1)^d \sim H_p[0,1)^d \sim H_p^i[0,1)^d \sim L_p[0,1)^d \qquad (1 < p < \infty).$$

The hardy spaces $H_p^\circ[0,1)^d$. To obtain some convergence results of the summability means over the diagonal we consider the Hardy space $H_p^\circ[0,1)^d$. Now the situation is similar to the one-dimensional case.

A function $a \in L_\infty[0,1)^d$ is a *cube p-atom* if

(a) supp $a \subset I$, $I \subset [0,1)^d$ is a dyadic cube,
(b) $\|a\|_\infty \leq |I|^{-1/p}$,
(c) $\int_I a(x)\, dx = 0$.

The basic result of atomic decomposition is the following one (see Weisz [73, 91]).

THEOREM 9.10 *A d-parameter martingale f is in $H_p^\circ[0,1)^d$ $(0 < p \leq 1)$ if and only if there exist a sequence $(a^k, k \in \mathbf{N})$ of cube p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a^k = f \quad \textit{in the sense of martingales,}$$
$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{9.8}$$

Moreover,

$$\|f\|_{H_p^\circ[0,1)^d} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p} \tag{9.9}$$

where the infimum is taken over all decompositions of f of the form (9.8).

For a rectangle $R = I_1 \times \ldots \times I_d \subset \mathbf{R}^d$ let $R^r := 2^r R := I_1^r \times \ldots \times I_d^r$ $(r \in \mathbf{N})$. The following result generalizes Theorem 9.2.

THEOREM 9.11 *Suppose that*

$$\int_{[0,1)^d \setminus I^r} |Va|^{p_0} \, d\lambda \le C_{p_0}$$

for all cube p_0-atoms a and for some fixed $r \in \mathbf{N}$ and $0 < p_0 \le 1$. If the sublinear operator V is bounded from $L_{p_1}[0,1)^d$ to $L_{p_1}[0,1)^d$ $(1 < p_1 \le \infty)$ then

$$\|Vf\|_p \le C_p \|f\|_{H_p^\circ} \qquad (f \in H_p^\circ[0,1)^d) \tag{9.10}$$

for all $p_0 \le p \le p_1$. Moreover, if $p_0 < 1$ then the operator V is of weak type $(1,1)$, i.e. if $f \in L_1[0,1)^d$ then

$$\sup_{\rho > 0} \rho \lambda(|Vf| > \rho) \le C \|f\|_1. \tag{9.11}$$

Again, (9.11) follows from (9.10) by interpolation.

The Hardy spaces $H_p[0,1)^d$. In the investigation of the convergence in the Prighheim's sense (i.e. over all n) we use the Hardy spaces $H_p[0,1)^d$. The atomic decomposition for $H_p[0,1)^d$ is much more complicated. One reason of this is that the support of an atom is not a rectangle but an open set. Moreover, here we have to choose the atoms from $L_2[0,1)^d$ instead of $L_\infty[0,1)^d$. This atomic decomposition was proved by Chang and Fefferman [9, 14] and Weisz [85, 91]. For an open set $F \subset [0,1)^d$ denote by $\mathcal{M}(F)$ the maximal dyadic subrectangles of F.

A function $a \in L_2$ is a *p-atom* if

(a) $\operatorname{supp} a \subset F$ for some open set $F \subset [0,1)^d$,

(b) $\|a\|_2 \le |F|^{1/2-1/p}$,

(c) a can be further decomposed into the sum of "elementary particles" $a_R \in L_2$, $a = \sum_{R\in\mathcal{M}(F)} a_R$ in L_2, satisfying

(d) $\operatorname{supp} a_R \subset R \subset F$,

(e) for all $i = 1, \dots, d$ and $R \in \mathcal{M}(F)$ we have

$$\int_{[0,1)} a_R(x)\, dx_i = 0,$$

(f) for every disjoint partition $\mathcal{P}_l$ $(l = 1, 2, \dots)$ of $\mathcal{M}(F)$,

$$\Big(\sum_l \| \sum_{R\in\mathcal{P}_l} a_R\|_2^2\Big)^{1/2} \le |F|^{1/2-1/p}.$$

THEOREM 9.12 *A d-parameter martingale f is in $H_p[0,1)^d$ $(0 < p \le 1)$ if and only if there exist a sequence $(a^k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\begin{aligned} &\sum_{k=0}^{\infty} \mu_k a^k = f \quad \textit{in the sense of martingales,} \\ &\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \end{aligned} \tag{9.12}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p\Big)^{1/p}$$

where the infimum is taken over all decompositions of f of the form (9.12).

The corresponding results to Theorems 9.2 and 9.11 for the $H_p[0,1)^d$ space are much more complicated. First we consider the two-dimensional case. Since the definition of the p-atom is very complex, to obtain a usable condition about the boundedness of the operators, we have to introduce simpler atoms.

If $d = 2$, a function $a \in L_2[0,1)^2$ is called a *simple p-atom*, if

(a) $\operatorname{supp} a \subset R$, $R \subset [0,1)^2$ is a dyadic rectangle,

(b) $\|a\|_2 \le |R|^{1/2-1/p}$,

(c) $\int_{[0,1)} a(x)\, dx_i = 0$ for $i = 1, 2$.

Note that $H_p[0,1)^d$ cannot be decomposed into rectangle p-atoms, a counterexample can be found in Weisz [73]. However, the following result says that for an operator V to be bounded from $H_p[0,1)^d$ to $L_p[0,1)^d$ $(0 < p \leq 1)$ it is enough to check V on simple p-atoms and the boundedness of V on $L_2[0,1)^d$.

THEOREM 9.13 *Suppose that $d = 2$, $0 < p_0 \leq 1$ and there exists $\eta > 0$ such that*

$$\int_{[0,1)^2 \setminus R^r} |Va|^{p_0}\, d\lambda \leq C_{p_0} 2^{-\eta r}, \tag{9.13}$$

for all simple p_0-atoms a and for all $r \geq 1$. If the sublinear operator V is bounded from $L_2[0,1)^d$ to $L_2[0,1)^d$, then

$$\|Vf\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)^d) \tag{9.14}$$

for all $p_0 \leq p \leq 2$. In particular, if $p_0 < 1$ then the operator V is of weak type $(H_1^i[0,1)^d, L_1[0,1)^d)$, i.e. if $f \in H_1^i[0,1)^d$ for some $i = 1, \ldots, d$ then

$$\sup_{\rho > 0} \rho \lambda(|Vf| > \rho) \leq C \|f\|_{H_1^i}. \tag{9.15}$$

Inequality (9.15) follows from (9.14) by interpolation. In some sense the space $H_1^i[0,1)^d$ plays the role of the one-dimensional $L_1[0,1)$ space.

Theorem 9.13 for two-dimensional classical Hardy spaces is due to Fefferman [14] and for martingale Hardy spaces to Weisz [79]. Journé [36] verified that the preceding result do not hold for dimensions greater than 2. So there are fundamental differences between the theory in the two-parameter and three- or more-parameter cases. Now we present the analogous theorem for higher dimensions.

Let $d \geq 3$. A function $a \in L_2[0,1)^d$ is called a *simple p-atom*, if there exist dyadic intervals $I_i \subset [0,1)$, $i = 1, \ldots, j$ for some $1 \leq j \leq d-1$ such that

(a) supp $a \subset I_1 \times \ldots I_j \times A$ for some measurable set $A \subset [0,1)^{d-j}$,

(b) $\|a\|_2 \leq (|I_1| \cdots |I_j||A|)^{1/2-1/p}$,

(c) $\int_{I_i} a(x) x_i \, dx_i = \int_A a \, d\lambda = 0$ for $i = 1, \ldots, j$.

Of course if $a \in L_2[0,1)^d$ satisfies these conditions for another subset of $\{1, \ldots, d\}$ than $\{1, \ldots, j\}$, then it is also called simple p-atom.

As in the two-parameter case, $H_p[0,1)^d$ cannot be decomposed into simple p-atoms. It is easy to see that condition (9.13) can also be formulated as follows:

$$\int_{(I_1^r)^c \times I_2} |Va|^{p_0}\, d\lambda + \int_{(I_1^r)^c \times I_2^c} |Va|^{p_0}\, d\lambda \leq C_{p_0} 2^{-\eta r}$$

and the corresponding inequality holds for the dilation of I_2, where H^c denotes the complement of the set H and $R = I_1 \times I_2$. For higher dimensions we generalize this form. The next theorem is due to the author [85, 91].

THEOREM 9.14 *Let $d \geq 3$. Suppose that the operators V_n are linear for every $n \in \mathbf{N}^d$ and*

$$V := \sup_{n \in \mathbf{N}^d} |V_n|$$

is bounded on $L_2[0,1)^d$. Suppose that there exist $\eta_1, \dots, \eta_d > 0$, such that for all simple p_0-atoms a and for all $r_1 \dots, r_d \geq 1$

$$\int_{(I_1^{r_1})^c \times \ldots \times (I_j^{r_j})^c} \int_A |Va|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_j r_j}.$$

If $j = d-1$ and $A = I_d \subset [0,1)$ is a dyadic interval, then we assume also that

$$\int_{(I_1^{r_1})^c \times \ldots \times (I_{d-1}^{r_{d-1}})^c} \int_{(I_d)^c} |Va|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_{d-1} r_{d-1}}.$$

Then

$$\|V^* f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)^d)$$

for all $p_0 \leq p \leq 2$. In particular, if $p_0 < 1$ and $f \in H_1^i[0,1)^d$ for some $i = 1, \dots, d$ then

$$\sup_{\rho > 0} \rho \lambda(|Vf| > \rho) \leq C \|f\|_{H_1^i}. \tag{9.16}$$

More-dimensional Walsh Functions

The Kronecker product $(w_n, n \in \mathbf{N}^d)$ of d Walsh systems is said to be the *d-dimensional Walsh system*. Thus

$$w_n(x) := w_{n_1}(x_1) \cdots w_{n_d}(x_d)$$

where $n = (n_1, \dots, n_d) \in \mathbf{N}^d$, $x = (x_1, \dots, x_d) \in [0,1)^d$.

The nth Fourier coefficient of $f \in L_1[0,1)^d$ is introduced by

$$\hat{f}(n) := \int_{[0,1)^d} f w_n \, d\lambda \qquad (n \in \mathbf{N}^d).$$

With the usual extension of Fourier coefficients to martingales we can define the nth partial sum of the Walsh-Fourier series of a martingale f by

$$s_n f := \sum_{j=1}^{d} \sum_{k_j=0}^{n_j-1} \hat{f}(k) w_k, \qquad (n \in \mathbf{N}^d).$$

Under $\sum_{j=1}^d \sum_{k_j=0}^{n_j-1}$ we mean the sum $\sum_{k_1=0}^{n_1-1} \dots \sum_{k_d=0}^{n_d-1}$.

It is known that $s_{2^{n_1},\dots,2^{n_d}} f = f_n$ $(n \in \mathbf{N}^d)$ and

$$s_{2^{n_1},\dots,2^{n_d}} f \to f \qquad \text{in } L_p\text{-norm as } n \to \infty,$$

if $f \in L_p[0,1)^d$ $(1 \le p < \infty)$. If $p > 1$ then the convergence holds also a.e. Moreover,

$$s_n f \to f \qquad \text{in } L_p\text{-norm as } n \to \infty,$$

whenever $f \in L_p[0,1)^d$ $(1 < p < \infty)$ (see e.g. Schipp, Wade, Simon and Pál [56]). The a.e. convergence of $s_n f$ is not true (Fefferman [12, 13]). However, investigating the partial sums over the diagonal, only, we have the following results (Móricz [42] or Schipp, Wade, Simon and Pál [56]):

$$\| \sup_{n \in \mathbf{N}} |s_{\mathbf{n}} f| \|_2 \le C \|f\|_2 \qquad (f \in L_2[0,1)^d)$$

and for $f \in L_2[0,1)^d$

$$s_{\mathbf{n}} f \to f \qquad \text{a.e. as } n \to \infty \ (n \in \mathbf{N}). \tag{9.17}$$

In contrary to the trigonometric case, it is unknown whether this result holds for functions in $L_p[0,1)^d$, $1 < p < 2$.

Summability of *d*-dimensional Walsh-Fourier series

The *Fejér*, *Cesàro* and *Riesz means* of a martingale f are defined by

$$\sigma_n f := \frac{1}{\prod_{i=1}^d n_i} \sum_{j=1}^d \sum_{k_j=1}^{n_j} s_k f = \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \prod_{i=1}^d \Big(1 - \frac{k_i}{n_i}\Big) \hat{f}(k) w_k,$$

$$\begin{aligned} \sigma_n^\alpha f \quad &:= \quad \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^d \sum_{k_j=1}^{n_j} A_{n_j-k_j}^{\alpha_j-1} s_k f \\ &= \quad \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \Big(\prod_{i=1}^d A_{n_i-k_i-1}^{\alpha_i} \Big) \hat{f}(k) w_k \end{aligned}$$

and

$$\sigma_n^{\alpha,\gamma} f := \frac{1}{\prod_{i=1}^d n_i^{\alpha_i \gamma_i}} \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \Big(\prod_{i=1}^d (n_i^{\gamma_i} - k_i^{\gamma_i})^{\alpha_i} \Big) \hat{f}(k) w_k,$$

respectively. We use again the common notation σ_n for the Cesàro and Riesz means. For a given $\tau \ge 0$ the *restricted and non-restricted maximal operators* are defined by

$$\sigma_\circ f := \sup_{\substack{2^{-\tau} \le n_i/n_j \le 2^\tau \\ i,j=1,\dots,d}} |\sigma_n f|, \qquad \sigma_* f := \sup_{n \in \mathbf{N}^d} |\sigma_n f|.$$

The next result follows easily from Theorem 9.4 by iteration.

THEOREM 9.15 *If* $0 < \alpha_j \leq 1 \leq \gamma_j$ $(j = 1, \ldots, d)$ *and* $1 < p \leq \infty$ *then*

$$\|\sigma_* f\|_p \leq C_p \|f\|_p \qquad (f \in L_p[0,1)^d).$$

Moreover, for all $f \in L_p[0,1)^d$ $(1 < p < \infty)$,

$$\sigma_n f \to f \qquad \text{a.e. and in } L_p\text{-norm as } n \to \infty.$$

The L_p-norm convergence holds also, if $p = 1$. Here $n \to \infty$ means that $\min(n_1, \ldots, n_d) \to \infty$ (the Pringsheim's sense of convergence).

Restricted summability. In this subsection we investigate the operator $\sigma_\circ$ and the convergence of $\sigma_n f$ over the cone $\{n \in \mathbf{N}^d : 2^{-\tau} \leq n_i/n_j \leq 2^\tau; i, j = 1, \ldots, d\}$, where $\tau \geq 0$ is fixed.

THEOREM 9.16 *If* $0 < \alpha_j \leq 1 \leq \gamma_j$ $(j = 1, \ldots, d)$ *and*

$$p_0 := \max\{1/(\alpha_j + 1), j = 1, \ldots, d\} < p < \infty,$$

then

$$\|\sigma_\circ f\|_p \leq C_p \|f\|_{H_p^\circ} \qquad (f \in H_p^\circ[0,1)^d).$$

This theorem for $\alpha_1 = \alpha_2 = 1$ and for two-dimensional functions was proved by the author [75]. The general version of Theorem 9.16 can be found in Weisz [84, 91] (see also Goginava [27]).

For the Fejér means (i.e. $\alpha_j = \gamma_j = 1$, $j = 1, \ldots, d$) there are counterexamples for the boundedness of $\sigma_\circ$ if $p \leq p_0 = 1/2$ (Goginava [33]).

THEOREM 9.17 *The operator* $\sigma_\circ^1$ $(\alpha_j = 1, j = 1, \ldots, d)$ *is not bounded from* $H_p^\circ[0,1)^d$ *to* $L_p[0,1)^d$ *if* $0 < p \leq 1/2$.

By interpolation we obtain ([84])

COROLLARY 9.6 *If* $0 < \alpha_j \leq 1 \leq \gamma_j$ $(j = 1, \ldots, d)$ *and* $f \in L_1[0,1)^d$ *then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_\circ f > \rho) \leq C \|f\|_1.$$

The set of the Walsh polynomials is dense in $L_1[0,1)^d$, so Corollary 9.6 imply the Walsh analogue of the Marcinkiewicz-Zygmund result.

COROLLARY 9.7 *If* $0 < \alpha_j \leq 1 \leq \gamma_j$ $(j = 1, \ldots, d)$ *and* $f \in L_1[0,1)^d$ *then*

$$\sigma_n f \to f \qquad \text{a.e.}$$

as $n \to \infty$ *and* $2^{-\tau} \le n_i/n_j \le 2^{\tau}$ $(i, j = 1, \ldots, d)$.

Note that this corollary is due to the author [75, 84], for Fejér means and for two-dimensional functions it can also be found in Gát [20].

The following results are known ([84]) for the norm convergence of $\sigma_n f$.

THEOREM 9.18 *If* $0 < \alpha_j \le 1 \le \gamma_j$ $(j = 1, \ldots, d)$ *and* $p_0 < p < \infty$*, then*

$$\|\sigma_n f\|_{H_p^\circ} \le C_p \|f\|_{H_p^\circ} \qquad (f \in H_p^\circ[0,1)^d)$$

whenever $2^{-\tau} \le n_i/n_j \le 2^{\tau}$ $(i, j = 1, \ldots, d)$.

COROLLARY 9.8 *If* $0 < \alpha_j \le 1 \le \gamma_j$ $(j = 1, \ldots, d)$, $p_0 < p < \infty$ *and* $f \in H_p^\circ$ *then*

$$\sigma_n f \to f \qquad \text{in } H_p^\circ\text{-norm}$$

as $n \to \infty$ *and* $2^{-\tau} \le n_i/n_j \le 2^{\tau}$ $(i, j = 1, \ldots, d)$.

Unrestricted summability. Now we deal with the operator σ_* and the convergence of $\sigma_n f$ as $n \to \infty$, i.e. $\min(n_1, \ldots, n_d) \to \infty$. The next result is due to the author ([79, 87, 85, 92]).

THEOREM 9.19 *If* $0 < \alpha_j \le 1 \le \gamma_j$ $(j = 1, \ldots, d)$ *and*

$$p_0 := \max\{1/(\alpha_j + 1), j = 1, \ldots, d\} < p < \infty,$$

then

$$\|\sigma_* f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)^d).$$

THEOREM 9.20 (GOGINAVA [33]) *The operator* σ_*^1 $(\alpha_j = 1, j = 1, \ldots, d)$ *is not bounded from* $H_p[0,1)^d$ *to* $L_p[0,1)^d$ *if* $0 < p \le 1/2$.

By interpolation we get here a.e. convergence for functions from the spaces $H_1^i[0,1)^d$ instead of $L_1[0,1)^d$.

COROLLARY 9.9 *If* $0 < \alpha_j \le 1 \le \gamma_j$ *and* $f \in H_1^i[0,1)^d$ $(i, j = 1, \ldots, d)$ *then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_* f > \rho) \le C \|f\|_{H_1^i}.$$

Recall that $H_1^i[0,1)^d \supset L(\log L)^{d-1}[0,1)^d$ for all $i = 1, \ldots, d$.

COROLLARY 9.10 *If* $0 < \alpha_j \le 1 \le \gamma_j$ *and* $f \in H_1^i[0,1)^d$ $(i, j = 1, \ldots, d)$ *then*

$$\sigma_n f \to f \qquad \text{a.e. as } n \to \infty.$$

Gát [23, 24] proved for the Fejér means that this corollary does not hold for all integrable functions.

THEOREM 9.21 *The a.e. convergence is not true for all $f \in L_1[0,1)^d$.*

THEOREM 9.22 *If $0 < \alpha_j \le 1 \le \gamma_j$ $(j = 1, \dots, d)$ and $p_0 < p < \infty$, then*

$$\|\sigma_n f\|_{H_p} \le C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)^d, n \in \mathbf{N}^d).$$

COROLLARY 9.11 *If $0 < \alpha_j \le 1 \le \gamma_j$ $(j = 1, \dots, d)$, $p_0 < p < \infty$ and $f \in H_p$ then*

$$\sigma_n f \to f \qquad \text{in } H_p\text{-norm as } n \to \infty.$$

5. More-dimensional Dyadic Derivative

The multi-dimensional version of Lebesgue's differentiation theorem reads as follows:

$$f(x) = \lim_{h \to 0} \frac{1}{\prod_{j=1}^d h_j} \int_{x_1}^{x_1+h_1} \dots \int_{x_d}^{x_d+h_d} f(t)\, dt \qquad \text{a.e.,}$$

if $f \in L(\log L)^{d-1}[0,1)^d$. If $\tau^{-1} \le |h_i/h_j| \le \tau$ for some fixed $\tau \ge 0$ and all $i, j = 1, \dots, d$, then it holds for all $f \in L_1[0,1)^d$. To present the dyadic version of this result we introduce first the *multi-dimensional dyadic derivative* ([4]) by the limit of

$$\begin{aligned}(\mathbf{d}_n f)(x) \quad := \quad & \sum_{i=1}^{d} \sum_{j_i=0}^{n_i-1} 2^{j_1+\dots+j_d-d} \\ & \times \sum_{\varepsilon_i=0}^{1} (-1)^{\varepsilon_1+\dots+\varepsilon_d} f(x_1 \dot{+} \varepsilon_1 2^{-j_1-1}, \dots, x_d \dot{+} \varepsilon_d 2^{-j_d-1}).\end{aligned}$$

The *d-dimensional dyadic integral* is defined by

$$\mathbf{I}f(x) := f * (W \times \dots \times W)(x) = \int_0^1 \dots \int_0^1 f(t) W(x_1 \dot{+} t_1) \cdots W(x_d \dot{+} t_d)\, dt$$

and for given $\tau \ge 0$ let the *maximal operators* be

$$\mathbf{I}_\circ f := \sup_{|n_i - n_j| \le \tau, i,j=1,\dots,d} |\mathbf{d}_n(\mathbf{I}f)|, \qquad \mathbf{I}_* f := \sup_{n \in \mathbf{N}^d} |\mathbf{d}_n(\mathbf{I}f)|.$$

THEOREM 9.23 *Suppose that $f \in H_p^\circ[0,1)^d \cap L_1[0,1)^d$ and*

$$\int_0^1 f(x)\, dx_i = 0 \quad (i = 1, \dots, d). \tag{9.18}$$

Then

$$\|\mathbf{I}_\circ f\|_p \leq C_p \|f\|_{H_p^\circ}$$

for all $d/(d+1) < p < \infty$.

COROLLARY 9.12 *If* $f \in L_1[0,1)^d$ *satisfies (9.18), then*

$$\sup_{\rho>0} \rho\, \lambda(\mathbf{I}_\circ f > \rho) \leq C\|f\|_1.$$

COROLLARY 9.13 *If* $\tau \geq 0$ *is arbitrary and* $f \in L_1[0,1)^d$ *satisfies (9.18), then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \quad \textit{a.e., as } n \to \infty \textit{ and } |n_i - n_j| \leq \tau.$$

Theorem 9.23 and Corollaries 9.12 and 9.13 are due to the author [76, 91]. The two corollaries were also shown by Gát [22].

For the operator $\mathbf{I}_*$ the following results were verified in Weisz [77, 88, 91].

THEOREM 9.24 *If (9.18) is satisfied and* $1/2 < p < \infty$ *then*

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)^d).$$

COROLLARY 9.14 *If* $f \in H_1^i[0,1)^d$ $(i = 1, \ldots, d)$ *satisfies (9.18), then*

$$\sup_{\rho>0} \rho\, \lambda(\mathbf{I}_* f > \rho) \leq C\|f\|_{H_1^i}.$$

COROLLARY 9.15 *If* $f \in H_1^i[0,1)^d (\supset L(\log L)^{d-1}[0,1)^d)$ $(i = 1, \ldots, d)$ *satisfies (9.18), then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \qquad \textit{a.e., as} \quad n \to \infty.$$

Note that this result for $f \in L \log L$ is due to Schipp and Wade [55] in the two-dimensional case.

Similarly to the dyadic derivative we can define the Vilenkin derivative (see Onneweer [46]) and one can prove similar results (see Pál and Simon [47, 48], Gát and Nagy [25] and Simon and Weisz [62, 61].

6. Marcinkiewicz-Cesàro summability of Walsh-Fourier Series

As we have seen in (9.17), the diagonal partial sums $s_{\mathbf{n}} f$ converge to f a.e. as $n \to \infty$. In this section we take the arithmetic means $\tau_n f$ (the so called Marcinkiewicz-Fejér means) of the sequence $(s_{\mathbf{n}} f)$ and present some a.e. results for $\tau_n f$ and inequalities.

Marcinkievicz [40] verified that the Marcinkiewicz-Fejér (or -Cesàro) means $\tau_n f$ of the two-dimensional trigonometric Fourier series of a function $f \in$

$L\log L(\mathbf{T}^2)$ converge a.e. to f as $n \to \infty$. Later Zhizhiashvili [96, 97] extended this result to all $f \in L_1(\mathbf{T}^2)$. Besides this convergence the author [86] proved that the maximal operator $\tau_* f := \sup_{n\in\mathbf{N}} |\tau_n f|$ is bounded from $H_p^\circ[0,1)^d$ to $L_p[0,1)^d$ and it is of weak type $(1,1)$, where $p_0 < p < \infty$ and $p_0 < 1$.

Let

$$\tau_n f := \tau_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_{\mathbf{k}} f$$

be the *Marcinkiewicz-Cesàro means* and

$$\tau_* f := \sup_{n\in\mathbf{N}} |\tau_n f|.$$

The following results were proved by Weisz [90, 91] and Goginava [28, 29, 30, 32].

THEOREM 9.25 *If* $0 < \alpha \le 1$ *and* $d/(d+\alpha) < p < \infty$ *then*

$$\|\tau_* f\|_p \le C_p \|f\|_{H_p^\circ} \qquad (f \in H_p^\circ[0,1)^d).$$

THEOREM 9.26 (GOGINAVA [31]) *The operator* τ_*^1 *is not bounded from* $H_p^\circ[0,1)^d$ *to* $L_p[0,1)^d$ *if* $0 < p \le d/(d+1)$.

COROLLARY 9.16 *If* $0 < \alpha \le 1$ *and* $f \in L_1[0,1)^d$ *then*

$$\sup_{\rho>0} \rho\lambda(\tau_* f > \rho) \le C\|f\|_1.$$

COROLLARY 9.17 *If* $0 < \alpha \le 1$ *and* $f \in L_1[0,1)^d$ *then*

$$\tau_n f \to f \qquad \textit{a.e. as } n \to \infty.$$

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Chapter 10

DISCRETE-TYPE RIESZ PRODUCTS

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Abstract We factorize finite data of length m, or step functions determined on the intervals $[k/m, (k+1)/m), k = 0, \ldots, m-1$ of $[0, 1)$, by writting them as a discrete Riesz-type Product $t_n = \prod_{k=1}^{m}(1 + a_k h_{k,n})$ with respect to the rows h_k of a matrix $H(m)$ of order $m \times m$ and associated to a sequence of coefficients $\{a_k : k = 1, \ldots, m\}$. We give sufficient conditions on $H(m)$ and $\{a_k\}$, providing invertibility of the underlying non-linear Riesz-type transform and we present examples of classes of acceptable matrices.

1. Introduction

The original Riesz's construction associated to a sequence of coefficients $\{a_n\}$, was to show that there exists a continuous function F of bounded variation in $[0, 2\pi)$, whose Fourier-Stieltjes coefficients do not vanish at infinity, F being the pointwise limit of the sequence of functions:

$$F_N(x) = \int_0^x \prod_{n=1}^{N}(1 + a_n \cos(2\pi 4^n t))dt.$$

Over the years, Riesz's construction was generalized, by replacing the generating function $cos(2\pi t)$ with other generating functions such as the Rademacher, or Walsh functions, or trigonometric polynomials (see [4], [6], [6]). Recently in [3], multiscale Riesz Products have been constructed, based on a real valued function H on $[0,1)$, called generating function and a dilation operator $T:[0,1)\to[0,1)$, such that:

$$\mu_m(\gamma)=\prod_{n=1}^{m}(1+a_nH(T^{n-1}\gamma))$$

converges weak-$*$ to a bounded measure as $m\to\infty$. Obviously, we can generalize the definition of μ_m, by considering partial Riesz Products of the form:

$$\mu_m(\gamma)=\prod_{n=1}^{m}(1+a_nH_n(\gamma)), \tag{10.1}$$

where $H_n(\gamma),(n=1,\dots,m)$ are bounded functions on $[0,1)$. Clearly, if we denote by V_m the space of sequences of length m and by $B[0,1)$ the space of bounded functions on $[0,1)$, the partial Riesz Products (10.1) induce a non-linear transform $\mu_m:V_m\to B[0,1)$, such that for every $a=\{a_1,\dots,a_m\}\in V_m$ we have:

$$\mu_m(a)(\gamma)=\prod_{n=1}^{m}(1+a_nH_n(\gamma)).$$

In order to achieve invertibility for μ_m, in [2] and [3] we considered step functions H_n on the intervals $\Omega_{n,m}=\left[\frac{n-1}{m},\frac{n}{m}\right)$, $n=1,\dots,m$:

$$H_n(\gamma)=\sum_{i=0}^{m}h_{n,i}\mathbf{1}_{\Omega_{\mathrm{i,m}}}(\gamma).$$

As a consequence, we dealt with discrete Riesz-type products of the form:

$$t_n=\prod_{k=1}^{m}(1+a_kh_{k,n}). \tag{10.2}$$

We proved the following:

THEOREM 10.1 *(see [3])*
Let $H(m)=\{h_{k,n}:k,n=1,\dots,m\}$ be a real orthonormal matrix whose first row is the constant row $(\frac{1}{\sqrt{m}},\dots,\frac{1}{\sqrt{m}})$ and all rows satisfy

$$h_nh_l=h_{n,l_0}h_l \quad \textit{whenever } n<l \tag{10.3}$$

where h_n,h_l are rows of $H(m)$ and h_{n,l_0} is the first non-zero entry of the l-row of the matrix $H(m)$. If $t=\{t_1,\dots,t_m\}$ is a sequence of real numbers such that

$$\langle t,h_i\rangle\neq 0,\ i=1,\dots,m,$$

where $<,>$ *is the usual inner product of* $\mathbf{R}^m$*, then there is a unique sequence of coefficients* $\{a_k : k = 1, \dots, m\}$ *such that:*

$$t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}). \tag{10.4}$$

Moreover, the coefficients $\{a_n : n = 1, \dots, m\}$ are computed via the following:

$$a_n = \begin{cases} \langle t, h_1 \rangle - \sqrt{m} & n = 1 \\ \frac{\langle t, h_n \rangle}{\prod_{k=1}^{n-1} (1 + a_k h_{k,n_0})}, & n = 2, \dots, m \end{cases},$$

where h_{n,n_0} is the first non-zero entry of the row h_n.

Also, we constructed a class of unbalanced Haar matrices $H(m)$ satisfying (10.3) of Theorem 10.1. An example is shown below:

$$H(3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$H(6) = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

In this paper, we relax the conditions imposed on the matrix $H(m)$ in Theorem 10.1. In Section 2, we see that Theorem 10.1 is true, if orthonormality is replaced by invertibility. Also, we show that for a particular class of exponential matrices we can drop (10.3) and Theorem10.1 is valid, as long as the values of the coefficients $\{a_k\}$ are restricted to the discrete set $A = \{0, 1\}$.

2. Discrete Riesz Products

In this section we obtain classes of matrices, whose corresponding Riesz Products give rise to an invertible non-linear transform.

PROPOSITION 10.1 *Let* $H(m) = \{h_{k,n} : k, n = 1, \dots, m\}$ *be a real invertible matrix satisfying (10.3) of Theorem 10.1. If* $t = \{t_1, \dots, t_m\}$ *is a sequence of real numbers such that*

$$\langle t, h_i \rangle \neq 0, \ i = 1, ..., m,$$

then there is a unique sequence of coefficients $\{a_k : k = 1, \dots, m\}$ *such that:*

$$t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}).$$

Moreover, the coefficients $\{a_n : n = 1, \dots, m\}$ *are computed via the following:*

$$a_n = \begin{cases} \langle t, h^{-1}_{\cdot,1} \rangle - \langle 1, h^{-1}_{\cdot,1} \rangle & n = 1 \\ \frac{\langle t, h^{-1}_{\cdot,n} \rangle}{\prod_{k=1}^{n-1} (1 + a_k h_{k,n_0})}, & n = 2, \dots, m \end{cases},$$

where $H^{-1}(m) = [h^{-1}_{j,k}]$ *is the inverse matrix of* $H(m)$*.*

Proof. We expand the discrete Riesz Product and we use (10.3) to get:

$$\begin{aligned} t_n &= 1 + \sum_{k=1}^{m} a_k h_{k,n} + \sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} a_{k_1} a_{k_2} h_{k_1,k_2^0} h_{k_2,n} + \dots \\ &\quad + (a_1 \dots a_m) \left(\prod_{j=1}^{m-1} h_{k_j,k_m^0} \right) h_{m,n}, \end{aligned}$$

where h_{k_j,k_m^0} is the first non-zero entry of the row h_{k_j}.

The invertibility of $H(m)$ and (10.4) imply that $\left\langle t, h^{-1}_{\cdot,1} \right\rangle = \left\langle 1, h^{-1}_{\cdot,1} \right\rangle + a_1$. For any $s > 1$ we have:

$$\begin{aligned} \left\langle t, h^{-1}_{\cdot,s} \right\rangle &= a_s \left(1 + \sum_{k_1=1}^{s-1} a_{k_1} h_{k_1,s_0} + \sum_{k_1=1}^{m-2} \sum_{k_2=k_1+1}^{m-1} a_{k_1} a_{k_2} \left(\prod_{j=1}^{2} h_{k_j,s_0} \right) \right. \\ &\quad \left. + \dots + (a_1 \dots a_{s-1}) \left(\prod_{j=1}^{s-1} h_{k_j,s_0} \right) \right) \\ &= a_s \prod_{k=1}^{s-1} (1 + a_k h_{k,s_0}). \end{aligned}$$

EXAMPLE 10.1 *A class of matrices* $H(m)$ *satisfying Proposition 10.1 is produced by the following rules:*

- *(a) The first row of* $H(m)$ *is the constant row* $\{1, \dots, 1\}$*.*
- *(b) Every other row has only two non-zero entries* 0 *or* 1*.*

(c) *If we denote by* $supp\{h_k\} = \{j \in \{1, \ldots, m\} : h_{kj} \neq 0\}$*, then:*

$$supp\{h_k\} \cap supp\{h_l\} = \oslash \text{ or } supp\{h_k\} \cap supp\{h_l\} = supp\{h_l\}$$

whenever $k < l$.

Below, we present examples of matrices satisfying rules (a)-(c):

$$\begin{aligned} H(3) &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ H(6) &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

PROPOSITION 10.2 *Let* $\Theta(m) = \{\theta_{n,j} : |\theta_{n,j}| < \pi,\ n, j = 1, \ldots, m\}$ *be an invertible matrix whose columns satisfy the following:*

$$-\pi \leq \sum_{n=1}^{m} \theta_{n,j} \leq \pi,\ j = 1, ..., m.$$

If $t = \{t_j = |t_j| e^{i \arg(t_j)}, j = 1, ..., m\}$ $-\pi \leq arg(z) \leq \pi$ *is a sequence of complex numbers, then there is a unique sequence of boolean coefficients* $\{a_n : n = 1, \ldots, m\}$*, such that:*

$$t_j = \prod_{n=1}^{m} (1 + a_n e^{i\theta_{n,j}}).$$

Moreover, the coefficients $\{a_n : n = 1, \ldots, m\}$ *are computed via the following matrix equation:*

$$a = 2\Theta^{-1} C(t),$$

where $a = [a_n]$ *and* $C(t) = [\arg(t_n)]$ *are column matrices of order* $m \times 1$.

Proof. Let $t_j = \prod_{n=1}^{m} (1 + a_n e^{i\theta_{n,j}})$, where $a_n \in \{0, 1\}$, then we have:

$$t_j = \prod_{n=1}^{m} (1 + a_n e^{i\theta_{n,j}}) = \prod_{n=1}^{m} (1 + e^{i\theta_{n,j}})^{a_n}.$$

Since

$$\overline{t_j} = \prod_{n=1}^{m} (1 + e^{-i\theta_{n,j}})^{a_n} = \prod_{n=1}^{m} \left(e^{-i\theta_{n,j}} \left(e^{i\theta_{n,j}} + 1 \right) \right)^{a_n}$$

$$= \prod_{n=1}^{m} \left(e^{-ia_n\theta_{n,j}} \left(1 + e^{i\theta_{n,j}}\right)^{a_n} \right) = t_j e^{-i\sum_{n=1}^{m} a_n\theta_{n,j}},$$

we get:

$$e^{-i\arg(t_j)} = e^{i\arg(t_j)} e^{-i\sum_{n=1}^{m} a_n\theta_{n,j}},$$

thus:

$$\sum_{n=1}^{m} a_n\theta_{n,j} = 2\arg(t_j) + 2\lambda_j\pi, \lambda \in \mathbf{Z}.$$

The hypothesis $-\pi \leq \sum_{n=1}^{m} \theta_{n,j} \leq \pi$ indicates that $\lambda_j = 0$ for every j and the result follows as a consequence of the invertibility of the matrix Θ.

EXAMPLE 10.2 *(Haar-type unbalanced matrices)*
Since Haar type unbalanced matrices $H(m)$ as defined in [3] have rows with zero mean, except for the first row which is the constant row $(\frac{1}{\sqrt{p^m}}, \ldots, \frac{1}{\sqrt{p^m}})$, orthogonal matrices of the form

$$\Theta(m) = \frac{\pi}{\sqrt{p^m}} H(m)$$

satisfy Proposition 10.2. We present below two examples:

$$\Theta(3) = \frac{\pi}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$H(6) = \frac{\pi}{\sqrt{6}} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

EXAMPLE 10.3 *(Generalized Walsh-type Riesz Products)*
Since Walsh orthogonal matrices $W(2^k), k = 1, \ldots,$ produced from the Walsh system $\{w_0, \ldots, w_{2^k}\}$ defined in [6] have rows with zero mean, except for the first row which is the constant row $(1, \ldots, 1)$, orthogonal matrices of the form

$$\Theta(2^k) = \frac{\pi}{2^k} W(2^k)$$

satisfy Proposition 10.2. We present below two examples:

$$\begin{aligned}\Theta(2) &= \frac{\pi}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ \Theta(4) &= \frac{\pi}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ldots\end{aligned}$$

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Chapter 11

A WALSH-TYPE MULTIRESOLUTION ANALYSIS

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Abstract We introduce a class of orthonormal matrices $U^{(n)}$ of order $p^n \times p^n$, $p = 2, \ldots, n = 1, \ldots$. The construction of those matrices is achieved in different scales by an iteration process, determined by a repetitive block matrix operation, involving the cross product of properly selected sub-matrices. For the case $p = 2$ we get the well known Walsh system. This particular construction also induces a multiscale transform on $L_2(\mathbf{T})$, reminiscent (although different) of a multiresolution analysis of $L_2(\mathbf{T})$.

1. Introduction

In order to provide efficient multiscale analysis on finite data, we seek for linear transforms whose corresponding matrices have the ability to detect specific characteristics from those data. In [4], we introduced a class of weighted sparse matrices for the purpose of prediction of almost periodic time series, while in [5] we built sparse matrices capable of revealing local information at different scales. In [3], we introduced a new class of sparse invertible matrices $H(m)$ of order $m \times m$, suitable for grammar detection of symbolic sequences. In fact, the matrices $H(m)$ may be considered as a generalization of the usual Haar matrices, since their construction was based on dilation and translation operations on unbalanced Haar functions. Thus, we obtained a generalized Haar transform:

$$\{t_n : n = 1, \ldots, m\} \leftrightarrow \{< t, h_n >: n = 1, \ldots, m\},$$

where $<, >$ is the usual inner product of the Euclidean space $\mathbf{R}^m$ and where h_n are the rows of $H(m)$.

In this work we dealt with the problem: what happens if we use dilation and replication operations, instead of using dilation and translation operations on matrices?

In Section 2, we build a discrete transform on finite data by using an iteration in scales. The cross product of matrices plays a central role in our construct, because it can be used either as a dilation or replication operator. So, we start from an initial matrix U of order $p \times p$. In every step of the iteration process we create a new matrix $U^{(n)}$ of order $p^n \times p^n$. $U^{(n)}$ is a block matrix, whose block sub-matrices are defined from the cross product $U^{(n-1)} \otimes U_i$ (see below). In Theorem 11.1, we prove that the matrices $U^{(n)}$ are orthonormal, whenever the initial matrix U is orthonormal. Thus, we obtain a discrete transform:

$$\{t_i : i = 1, \dots, p^n\} \leftrightarrow \{< t, U_i^{(n)} >: i = 1, \dots, p^n\},$$

where $U_i^{(n)}$ are the rows of $U^{(n)}$. For a suitable selection of the matrix U we see that the resulting orthonormal system is the Walsh system.

Since to any row of the matrix $U^{(n)}$ there corresponds a step function on $\mathbf{T}$, an orthonormal set $\widetilde{M}_n = \{\widetilde{m}_k(\gamma) : k = 1, \dots, p^n\}$ of functions of $L_2(\mathbf{T})$ emerges naturally from the matrix $U^{(n)}$. In Section 3 we see that the set $\widetilde{M}_n$ is produced by successive dilations and replicas of a generator set of functions $M = \{m_i(\gamma) : i = 0, \dots, p-1\}$:

$$m_i(\gamma) = \sum_{j=1}^{p} U_{i+1,j} \mathbf{1}_{\left[\frac{j-1}{p}, \frac{j}{p}\right)}, \; i = 0, \dots, p-1.$$

Indeed:

$$\widetilde{M}_n = \left\{ \widetilde{m}_k(\gamma) = \prod_{j=0}^{n-1} m_{\varepsilon_j}(p^j\gamma) : \; k = 1 + \sum_{j=0}^{n-1} \varepsilon_j p^j, \; \varepsilon_j \in \{0, \dots, p-1\} \right\}.$$

Finally, we see that our multiscale construction naturally extends to an invertible transform on $L_2(\mathbf{T})$.

2. A class of Walsh-type discrete transforms

Notation: Let $M_{n,m}$ be the set of all matrices of order $n \times m$ over the field of complex numbers. If $n = m$, then $M_{n,m}$ is abbreviated to M_n. We shall use the symbolism $A = [A_{ij}]$ to denote a matrix A with elements A_{ij}. The notation

$$A_i = \{A_{i,j} : j = 1, \dots, m\}$$

shall be used to denote the i-row of a matrix A. We define the following operators:

DEFINITION 11.1 *For $p = 2, \ldots,$ the tensor product of two matrices $A \in M_{n,m}$ and $B \in M_{k,l}$ is a block matrix $A \otimes B \in M_{nk,ml}$:*

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix}.$$

DEFINITION 11.2 *Let $S : M_{n_1,m} \times \ldots \times M_{n_k,m} \to M_{n_1+\ldots+n_k,m}$ be the following block matrix operator:*

$$S(M_1, \ldots, M_k) = \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix}.$$

DEFINITION 11.3 *Let U be an orthonormal matrix of order $p \times p$, we define a sequence of block matrices $U^{(n)}$, where $n = 1, ..., N$, by using the following iteration:*

$$U^{(n)} = \begin{cases} U, & n = 1 \\ S\left(U^{(n-1)} \otimes U_1, \ldots, U^{(n-1)} \otimes U_p\right), & n = 2, \ldots, N \end{cases}, \tag{11.1}$$

where U_i is the i row of U.

THEOREM 11.1 *The matrix $U^{(n)}, (n = 1, \ldots, N)$ is orthonormal.*

Proof. We work inductively. Clearly, the theorem is true for $n = 1$. We suppose that the matrix $U^{(n-1)}$ is orthonormal, so it suffices to prove that $< U_j^{(n)}, U_l^{(n)} >= \delta_{j,l}$, where $<,>$ is the usual inner product of the Euclidean space $\mathbf{R}^{p^n}$. Let $j = mp^{n-1} + \zeta,\ l = qp^{n-1} + \sigma$, where $m, q = 0, \ldots, p-1,\ \zeta, \sigma = 1, \ldots, p^{n-1}$, then:

$$\begin{aligned} \langle U_j^{(n)}, U_l^{(n)} \rangle &= \sum_{r=1}^{p^n} U_{jr}^{(n)} U_{rl}^{(n)} = \sum_{\nu=0}^{p-1} \sum_{\mu=1}^{p^{n-1}} U_{j,\nu p^{n-1}+\mu}^{(n)} U_{\nu p^{n-1}+\mu,l}^{(n)} \\ &= \sum_{\nu=0}^{p-1} \sum_{\mu=1}^{p^{n-1}} U_{m+1,\nu+1} U_{\zeta,\mu}^{(n-1)} U_{\nu+1,q+1} U_{\mu,\sigma}^{(n-1)} \\ &= \left(\sum_{\nu=1}^{p} U_{m+1,\nu} U_{\nu,q+1} \right) \left(\sum_{\mu=1}^{p^{n-1}} U_{\zeta,\mu}^{(n-1)} U_{\mu,\sigma}^{(n-1)} \right) \\ &= \delta_{m,q} \delta_{\zeta,\sigma} = \delta_{j,l}. \end{aligned}$$

It is clear that the inverse matrix of $U^{(n)}$ coincides to its transpose $\left(U^{(n)}\right)^T$. The following multiresolution structure arises from the matrices $U^{(n)}$:

Let V_{p^n} be the space of all real-valued sequences of length p^n and let $U_i^{(n)}$ be the i-row of the matrix $U^{(n)}$, then any element $t \in V_{p^n}$ can be written as:

$$t_l = \sum_{i=1}^{p^n} \langle t, U_i^{(n)} \rangle U_{i,l}^{(n)}.$$

Let $j = 1, \ldots, n-1,\ k = 1, \ldots, p-1$, we define

$$W_{j,k} = \mathrm{span}\{U_{kp^j+s}^{(n)} : s = 1, \ldots, p^j\},$$

then, we have the decomposition:

$$V_{p^n} = V_0 \oplus_{j=1}^{n-1} \oplus_{k=1}^{p-1} W_{j,k},$$

where $V_0 = \mathrm{span}\{U_s^{(n)} : s = 1, \ldots, p\}$.

EXAMPLE 11.1 *Let* $p = 3,\ n = 3$, *then* $V_{3^3} = V_0 \oplus_{j=1}^{2} \oplus_{k=1}^{2} W_{j,k}$, *where:*

$$V_0 = span\{U_1^{(3)}, \ldots, U_3^{(3)}\},$$

$$W_{1,1} = span\{U_4^{(3)}, \ldots, U_6^{(3)}\}, \quad W_{1,2} = span\{U_7^{(3)}, \ldots, U_9^{(3)}\},$$
$$W_{2,1} = span\{U_{10}^{(3)}, \ldots, U_{18}^{(3)}\}, \quad W_{2,2} = span\{U_{19}^{(3)}, \ldots, U_{27}^{(3)}\}.$$

DEFINITION 11.4 *Let* $p \geq 2$, *we define the following matrix* $\Psi^{(p)}$ *of order* $p \times p$*:*

$$\psi_{ij}^{p} = \begin{cases} \frac{1}{\sqrt{p}}, & \text{whenever } i = 1 \\ \frac{1}{\sqrt{p-i+1}} \frac{1}{\sqrt{p-i+2}}, & \text{whenever } 1 \leq j \leq p-i+1 \\ -\frac{\sqrt{p-i+1}}{\sqrt{p-i+2}}, & \text{whenever } j = p-i+2, \\ 0, & \text{whenever } p-i+2 < j \leq p \end{cases} \quad i, j = 1, \ldots, p.$$

EXAMPLE 11.2 $\Psi^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \Psi^3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\sqrt{2} \\ \frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} & 0 \end{pmatrix}.$

PROPOSITION 11.1 *(see [2])*
The matrix $\Psi^{(p)}$ *satisfies the following properties:*

(i) $\sum_{j=1}^{p} \psi_{ij}^{(p)} = 0,\ i = 2, \ldots, p.$

(ii) $\psi_i^{(p)}\psi_j^{(p)} = \psi_{i,1}^{(p)}\psi_j^{(p)}$, *whenever* $i < j, i, j = 1, \ldots, p$.

(iii) *The matrix* $\Psi^{(p)}$ *is orthonormal.*

OBSERVATION 11.1 *If we consider the iteration (11.1) with initial matrix* $U = \Psi^{(2)}$*, then we obtain the Walsh system (see [6]). Indeed:*

$$U^{(1)} = \Psi^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, U^{(2)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ldots.$$

Whenever $p > 2$*, we get a Walsh-type construction.*

OBSERVATION 11.2 *If we consider the iteration (11.1) with initial matrix* $U = \Psi^{(p)} = \left(e^{2\pi ikl/p}\right)_{k,l=0}^{p-1}$*, then we obtain the Generalized Walsh system as defined in [6].*

3. A multiscale transform on $L_2(\mathrm{T})$

We denote by $\mathbf{R}$ the additive group of real numbers and by $\mathbf{Z}$ the subgroup consisting of the integers. The group $\mathbf{T}$ is defined as the quotient $\mathbf{R}/\mathbf{Z}$. Since there is an obvious identification between functions on $\mathbf{T}$ and 1-periodic functions on $\mathbf{R}$, from now on we identify the elements of the space $L_2(\mathbf{T})$ of all complex valued Lebesgue square integrable functions on $\mathbf{T}$, as 1-periodic functions on $\mathbf{R}$.

Since any row $U_k^{(n)}$, $k = 1, \ldots, p^n$ of the matrix $U^{(n)}$ defined in Theorem 11.1 can be assigned to a step function $\widetilde{m}_k(\gamma)$ on $\mathbf{T}$ such that

$$\widetilde{m}_k(\gamma) = m_{kj},\ \gamma \in \Omega_{j,n} = \left[\frac{j-1}{p^n}, \frac{j}{p^n}\right),\ j = 1, \ldots, p^n,$$

an orthonormal set of functions of $L_2(\mathbf{T})$ emerges naturally from the construction presented in section 2:

$$\widetilde{M}_n = \left\{\widetilde{m}_k(\gamma) : \widetilde{m}_k(\gamma) = \sum_{j=1}^{p^n} U_{k,j}^{(n)} \mathbf{1}_{\Omega_{j,n}}(\gamma),\ k = 1, \ldots, p^n.\right\}$$

Moreover, if U is the initial orthonormal matrix of the iteration process (11.1), by defining:

$$m_i(\gamma) = \sum_{j=1}^{p} U_{i+1,j} \mathbf{1}_{\Omega_{j,n}}(\gamma),\ i = 0, \ldots, p-1,$$

we can see that the set $\widetilde{M}_n$ can be produced by successive dilations of the functions $m_i(\gamma)$ in the following:

$$\widetilde{M}_n = \left\{ \widetilde{m}_k(\gamma) = \prod_{j=0}^{n-1} m_{\varepsilon_j}(p^j\gamma),\ k = 1 + \sum_{j=0}^{n-1} \varepsilon_j p^j,\ \varepsilon_j \in \{0, \dots, p-1\} \right\}. \tag{11.2}$$

Moreover, we can prove:

THEOREM 11.2 *Let* $\{V_n : V_n \subset V_{n+1},\ n \geq 1\}$ *be a nested sequence of* p^n*-dimensional subspaces of* $L_2(\mathbf{T})$*, whose orthonormal basis is the set* $\widetilde{M}_n$ *defined in (11.2), then:*

$$\overline{\cup_{n\geq 1} V_n} = L_2(\mathbf{T}).$$

Proof. See [1].

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Chapter 12

GIBBS DERIVATIVES
40 YEARS AFTER THE INTRODUCTION OF THE CONCEPT

Notion, extensions, and generalizations - a brief overview

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Abstract In this paper we present a short account of the development of the theory of Gibbs differentiation on the occasion of 40th anniversary of its introduction [20], and the 35th of Butzer-Wagner dyadic derivative [10] as well as the 30th anniversary of Onneweer's extension of the theory to Vilenkin groups [40], the two most important results in two different ways of generalization of the Gibbs' introductory result. We also attempt to give a rather general characterization of Gibbs derivatives viewed as a class of differential operators on different not necessarily Abelian groups.

1. Background and motivation

Signals are physical conveyors of information. They are physical processes which spread in space-time and, therefore, are conveniently modeled by elements of some functional spaces. In this setting, signals are frequently identified with their mathematical models.

The following classes of signals can be distinguished in signal theory [45].

1. Signals modeled by real variable functions are called *continuous signals*. The continuous signals of the continuous amplitude are *analog* or *analogue signals*.

2. *Discrete signals* are modeled by discrete functions, i.e., they are usually considered as functions on the set of integers Z or on some of its subsets Z_q of integers less than some given q.

3. Discrete signals taking their values in some finite sets are *digital signals*.

Irrespective to which class belong, signals should be processed in order to extract, interpret, and exploit the information contained in them. Mathematics provide a theoretical base for disclosing various signal processing tools. The *Fourier analysis* and *differential calculus* are certainly the two most powerful among them.

The Fourier analysis expresses two important principles, the *superposition* and the *linearity*, both principles often present in engineering considerations of physical phenomena. In this way, the Fourier analysis provides means for mathematical description of a system in an approach frequently used in engineering practice, the decomposition of a complex problem into finitely or countably many simpler subproblems.

In the same ground, differential operators are conveniently used to express the direction as well as the rate of change of a signal at the input and/or output of a system, providing in this way the information about the state of the system considered.

A conviction prevalent for a long time was that signals existing in reality could be described adequately exclusively by functions on the real line R. A reason for that could be the conjecture that the topology of space-time is well represented by the topology of the real line. For that reason the Fourier analysis and differential calculus were first established, and for many years restricted, almost ultimately to R, the continuum which is one of the most sophisticated structures in mathematics.

This fact has been noticed as a *paradox of history* [25]. The mentioned conviction was greatly changed by the recognition of the so-called sampling theorem [36], [56], which states that, under certain conditions, continuous signals could be adequately represented by their discrete counterparts.

These conditions are conveniently expressed in terms of Fourier coefficients, and we can realize again the importance of Fourier analysis in signal processing.

It could be stated that the interest in practical engineering applications of discrete structures and functions defined on them sprang after the publication of famous Shannon's paper [56], although the essence of sampling theorem were known much earlier, see for example, [31].

In this settings the practical applicability of Fourier analysis were further supported by the rediscovery of the *Fast Fourier transform*, FFT, [16] which is a fast algorithm for the calculation of Fourier coefficients on finite groups. We use the term rediscovered since, similarly as in the case of sampling theorem, the essence of FFT was known to some authors of earlier times, as it is well documented in [30]. Today, FFT should be appreciated as a corner stone of the theory of fast signal processing transformations and the key part of many signal processing algorithms.

Among discrete functions, the *switching functions* defined as the mapping $f : \{0,1\}^n \rightarrow \{0,1\}$, $n \in N$ - the set of natural numbers, are probably most widely used, since all digital devices at the hardware level are realized by circuitry based upon two stable state basic elements.

Actually, the *switching theory* dealing with switching functions, provides a mathematical base for the description of behavior and functioning of digital devices and the representation of signals by binary sequences.

The *discrete Walsh transform* [73] is the basis for the Fourier analysis compatible with switching functions, the *Walsh-Fourier analysis*, since these functions can be conveniently viewed as a particular subset of functions on *finite dyadic group*, which consists of the set of 2^n binary n-tuples $x = (x_1, \ldots, x_n)$, $x_i \in \{0,1\}$, under the componentwise addition modulo 2.

Recall that the *Walsh functions*, the basis for the Walsh-Fourier analysis, are the group characters of the dyadic group [17], and therefore the Fourier analysis on that group is based upon them in the same way as the classical Fourier analysis is based upon the exponential functions e^{jwx}, the group characters of the real line R viewed as a particular locally compact Abelian group [53]. It is the same for the discrete Walsh functions viewed as the group characters of finite dyadic groups [1], [2].

The Walsh functions take only two different values ± 1 and, therefore are also in that respect compatible with binary-valued switching functions. This fact ensures at the same time the simplicity of computation with Walsh functions. As we noted, the Walsh transform is a particular case of Fourier transform on groups, and therefore, can be performed by fast transform algorithms derived as a particular case of FFT, see for example [1], [3], [54]. The fast Walsh transform can be computed without multiplication, which make it the computationally most efficient among Fourier transforms on different groups, since the multiplication is usually a more time consuming operation than addition when realized with present software and hardware technological resources.

In this way, the group theoretic approach to Fourier analysis, which suggests to use group characters for locally Abelian and unitary irreducible group representations for compact non-Abelan groups as kernels for the Fourier transform, enables a unique way of the extension of this theory to structures other than R and in particular to discrete structures.

Regarding differentiation, the extension was not so straightforward. The discrete and piecewise constant functions used with digital devices cannot be differentiated in the *Newton-Leibniz* sense and the need for an appropriate differential operator was apparent from almost the beginning of the use of discrete functions in engineering practice, especially in communications [39], [50].

It is quite understandable that first results were set in switching theory by the introduction of the *Boolean difference* [39], [50], since at the hardware level, digital communications are implemented through binary valued sequences.

The theory of this operator was established in [2],[69] and its application proved very useful in many areas of logic design and digital communications.

The term *Boolean differential calculus* is now often used, see for example explanation given in [70], although the Boolean difference can hardly be accepted as a proper differentiator, since it does not permit to distinguish the change of a switching variable from 0 to 1 from that of 1 to 0.

In any way, the Boolean difference is acting on the set of switching functions and, therefore, is a very particular answer to the problem of differentiation of discrete and piecewise constant functions used with digital devices.

The interest in Walsh functions, which raised in latest sixties, provide another motive for investigations on differentiation of such functions. At that time it was apparent a desire to consider the Walsh functions as a particular case of *special functions* [4], examples of which are Bessel, Chebyshev, Laguerre, Hermite, Lagrange, Legendre, etc., [4].

Special functions are usually generated as the solutions of some generating differential equations and, therefore, a corresponding differential operator was needed.

In the pioneering work in 1967, J. Edmund Gibbs proposed the following definition by attempting to answer to this desire for a differential operator in the case of discrete Walsh functions.

DEFINITION 12.1 *(Gibbs finite derivative)*
For a function f defined on finite dyadic group of order 2^n, the finite dyadic derivative $f^{(1)}$ is defined as

$$f^{(1)}(x) = -2^{-1}\sum_{r=0}^{n-1}(f(x \oplus 2^r) - f(x))2^r, \quad \forall x \in \{0, \ldots, 2^n - 1\}.$$

The finite dyadic derivative of a function f we also denote by Df. This operator D, also called *logical derivative* [23], [24], has the discrete Walsh functions as its eigenfunctions.

It follows that the discrete Walsh functions emerge as the solutions of the first order dyadic differential equation

$$Df - \lambda f = 0, \quad \lambda \in \{0, 1, \ldots, 2^n - 1\}.$$

As it is explained in [21], the operator was introduced quite independently on any work on Boolean difference. However, by relating these two operators [18], [19], the application of the finite dyadic derivative in the same areas where Boolean difference was already applied proved very interesting [32]. At the same time, the interpretation of finite dyadic derivative as the linear combination of partial dyadic derivatives, similar to Boolean differences relative to particular coordinates [64], offered a mean for the derivation of fast algorithms for the calculation of Gibbs derivatives on finite groups, the later generalizations of finite dyadic derivative [66]. Note that the term Gibbs derivatives was established and confirmed in particular at the *First international Workshop on Gibbs derivatives* in 1989 [9] in order to denote a broad family of differential operators representing the generalizations and extensions or were inspired by the finite dyadic derivative originated by J. Edmund Gibbs [21].

2. Generalizations of the finite dyadic Gibbs derivative

The great interest in practical engineering applications of Walsh analysis which sprang after the publication of Harmuth's paper in 1960 [29] provided a very suitable environment for further work on Gibbs derivatives. The activity in the area of Walsh analysis is best illustrated by the fact that from 1970 to 1974 the specialized conferences completely devoted to Walsh and related functions and their applications were organized in U.S.A. Moreover, in 1971, 1973, and 1975 the conferences on this particular subject were organized also in England, so that two conferences per year on the same subject were organized. This research work intended towards applications in computer science and engineering, leaded to the extension of the theory of Walsh and related functions into the so-called *spectral techniques* [5], [33], [34], [38], which are from that time the subject of specialized workshops or are discussed at standard sections at many conferences and meetings on signal processing and multiple-valued logic. Research papers on these subjects are accepted by many mathematical and engineering journals. For example, the journal *IEEE Transactions on Electromagnetic Compatibility* published by IEEE Press had the Associate Editors for Walsh and non-sinusoidal functions. This position was served very successfully by Henning F. Harmuth for many years.

The great activity in the area of Walsh functions have resulted among other things in some interesting generalizations and extensions of finite dyadic derivative. It should be noticed that 40 papers on the subject were presented at the conferences on Walsh and related functions from 1970 to 1975. Today the bibliography on Gibbs derivatives consists of over 277 items published by 69 authors from 14 nations all over the World. Among these, the probably most important and certainly most widely discussed result is the *Butzer-Wagner dyadic derivative* introduced 35 years ago [10].

Roughly speaking, two different ways of generalization of finite dyadic derivative can be distinguished. In essence, they are based upon two alternative interpretations of the basic property of finite dyadic derivative that the discrete Walsh functions are the eigenfunctions of that operator.

First, this property implies that the Walsh functions are infinitably many times dyadicaly differentiable. Various extensions of the finite dyadic Gibbs derivative were aimed at extending the class of differentiable functions.

The second implication concerns to the relationship of finite dyadic derivative with Walsh transform similar to the relationship of the Newton-Leibniz derivative with the Fourier transform in classical analysis on R. This was a basis for generalizations of Gibbs differentiation to groups other than the dyadic group.

Extension of the class of differentiable functions

The way of generalizations based upon the first implication, that started 35 years ago by Butzer and Wagner [10], is devoted to the extension of the class of functions differentiable in some sense, in this case, the dyadic sense. The approach were originated in Walsh-Fourier analysis on $[0, 1]$ and, therefore, mainly concerns functions on that interval. Recall that this interval can be identified with the infinite dyadic group consisting of countably many copies of the finite dyadic group of order 2 enriched with the product topology owing to the mapping

$$\lambda(x_1, x_2, \ldots) = \sum_{i=1}^{\infty} x_i 2^{-i}, \quad x_i \in \{0, 1\}.$$

The Walsh functions, being the characters of the dyadic group [17], form a complete orthonormal system in the space L^2 of measurable functions square integrable on the interval $[0, 1]$.

Butzer and Wagner [10] extended the concept of dyadic differentiation from finite dyadic group to the infinite dyadic group, or alternatively interval $[0, 1]$, by introducing a derivative D on $[0, 1]$ which eigenfunctions are the Walsh functions in the *Kaczmarcz ordering*, i.e., for which

$$D(wal_k) = k \cdot wal_k, \quad k = 0, 1, \ldots, \qquad (12.1)$$

where wal_k denotes the Walsh function of order k in the Kaczmarz ordering.

Butzer and Wagner also defined a derivative which expresses the same relation with respect to the *Paley ordered* Walsh functions [11].

Denoting by X either the space ϕ of functions continuous on $[0,1]$ or one of the spaces L^p, $1 < p < \infty$ of measurable functions whose p-th power is integrable over the interval $[0,1]$, the Butzer-Wagner derivative for *Paley ordered Walsh functions* can be described as follows.

DEFINITION 12.2 *(Butzer-Wagner derivative for Paley ordering)*
For a function $f \in X$ for which the sequence of functions

$$d_n(f,x) = \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x \oplus 2^{-j-1})),$$

converges in the norm of X the strong Butzer-Wagner dyadic derivative *is defined as the limit*

$$\lim_{n\to\infty} d_n(f,x).$$

Notice that Butzer and Wagner [13] further introduced the concept of the *pointwise dyadic derivative* by saying that a function from $[0,1]$ has the pointwise dyadic derivative at a point $x \in [0,1]$ if the sequence of real numbers $\{d_n(f,x)\}$ converges as $n \to \infty$.

The relation (12.1) is true for all $x \in [0,1]$ also for the pointwise Butzer-Wagner dyadic derivative with respect to the generalized Walsh functions $\Psi_y(x)$, i.e.,

$$D(\Psi_y)(x) = |y|\Psi_y(x),$$

for x, y taking values in the dyadic field.

As it is noticed in [6], [71], the dyadic derivative was especially adopted to functions having many jumps and possessing just a few and also short intervals of constancy. Even functions having a denumerable set of discontinuities like the well-known *Dirichlet function* can be dyadic differentiated on $[0,1]$ [76].

The extended dyadic derivative [7], [8] based upon the works by Butzer and Wagner [10], [11], [13] and He Zelin [77] is applicable also to piecewise polynomial functions, i.e., to functions which are made up entirely of polynomial pieces between the consecutive jumping points.

Extensions to functions on different groups

A possible characterization of Butzer-Wagner dyadic derivative can be given in terms of Walsh series coefficients $S_f(w)$ of f as

$$S_{Df}(w) = wS_f(w), \quad w = 1, 2, \ldots, \tag{12.2}$$

which is a consequence of the fact that Walsh functions are the eigenfunctions of this differential operator. Obviously, the same is true for dyadic derivative on finite groups [25].

The second way of generalization of the finite dyadic derivatives devoted to the transfer of the concept of differentiation to structures other than real line R is based just upon that characterization of finite dyadic derivative and the Butzer-Wagner dyadic derivatives.

The basic idea is relatively simple. The Newton-Leibniz derivative can be viewed as the linear operator mapping the exponential functions e^{jwx}, the characters of the real line R, into jw times of themselves.

The similar holds for the dyadic derivatives, relation (12.1), where the real line is replaced by the dyadic group and the exponential functions by the Walsh functions, the characters of the dyadic group. This relation reads as the relation (12.2) in the transform domain owing to the orthogonality of Fourier transform on groups. The theory should be extended to other locally compact Abelian or compact non-Abelian groups by the replacement of Walsh functions by the characters of the corresponding Abelian group or by unitary irreducible representations of the non-Abelian groups.

It is interesting to note that from this group-theoretic approach point of view there is apparent some parallelism between the ways of development of abstract harmonic analysis and the theory of Gibbs differential calculus on groups.

Recall that the abstract harmonic analysis is the mathematical discipline developed from the classical Fourier analysis by the replacement of the real line R, which is a particular locally compact Abelian group, by arbitrary locally compact Abelian or compact non-Abelian group. The same has been done in the case of Gibbs derivatives by using the relation (12.2) as a defining relationship of these operators. The first attempts in this direction were given again by J.E. Gibbs and his associates [14], [25].

A considerable extension of Gibbs differentiation on groups were given 30 years ago by Cornelis W.Onneweer [40] who introduced a Vilenkin group analogue of the dyadic derivative showing that the characters of the Vilenkin group are the eigenfunctions of the introduced differential operator and argued that the Butzer-Wagner characterization (12.2) caries over with extra work to this setting. See, also [49], [71].

In the similar way, there have been defined L^r-weak p-adic derivative, the adjacent p-adic derivative, the partial p-adic derivative [51], [74], [78]. See also [75].

Several other authors consider this way of generalizations of Gibbs differential calculus to other structures including also the discrete structures.

For example, Pál [46] defined the dyadic derivative Df on the dyadic field, i.e., for functions $f \in L^1(0, \infty)$ and showed that the Walsh transform F defined in [17], interacts with D as follows: if Df exists then $F(Df)(y) = yF(y)$ and $D(Fy)(x) = F(xf(x))$ if $xf(x) \in L^1(0,\infty)$.

In [48], Pál constructed an indefinite integral for D and proved a fundamental theorem of calculus in this setting. The extension of the dyadic differentiation to R+ were also considered, see for example [12], [47], as well as to local fields [44].

Recently, Golubov [27] introduced the modified strong dyadic integral J_α and the fractional derivative $D^{(\alpha)}$ of order $\alpha > 0$ for functions from the Lebesgue space $L(R_+)$. Established are criteria for existence of these integrals and derivatives for a given function $f \in L(R_+)$ and determined a countable set of eigenfunctions of these operators.

For the fractional dyadic derivative and integral, proven are in [28], the theorem on differentiation of the indefinite Lebesgue integral of an integrable function at its Lebesgue points, and the theorem on reconstruction of an absolutely continuous function by means of its derivative. These theorems can be viewed as analogues of the theorems of Lebesgue in classical analysis.

A class of generalizations of Gibbs differential calculus concerning both functions defined on the interval $[0,1]$ and on different discrete Abelian groups is obtained through the replacement of group characters by some other orthogonal systems, as for example, the system of Haar functions [58], discrete Haar functions [67] and generalized Haar functions [68], by an arbitrary orthogonal system [62], or even by an arbitrary bi-orthogonal system [59].

The transfer of the notion of Gibbs differentiation to finite non-Abelian groups was done in [60] and further considered in [61], [63], [65]. An approach to the extension of definition of Gibbs derivative to infinite non-Abelian groups was suggested in [61] following the idea of Butzer-Wagner definition of strong dyadic derivative.

3. Towards a general characterization of Gibbs derivatives

In this section we will attempt to give a characterization of Gibbs derivatives through a group-theoretic approach to the subject.

In order to cover in a uniform way the case of functions on Abelain and non-Abelian groups, we restrict the considerations to the space $K(G)$ of functions defined on a locally compact Abelian or a finite non-Abelian group G taking the values in a field K admitting the existence of a Fourier transform.

In a general ground, the Gibbs derivative of order k of a function $f \in K(G)$, which we denote by $D^k f$ is considered as the linear operator in $K(G)$ satisfying the relationship

$$(F(D^k f))(w) = \phi(w,k)(F(f))(w), \tag{12.3}$$

where F denotes the Fourier transform operator in $K(G)$.

In the most examples $\phi(w,k) = w^k$, but in some cases a scaling factor should be added, see, for example [23], while in a few particular cases the function ϕ differs and is related to the order of group G. For example, in the case of extended Butzer-Wagner dyadic derivative [7], $\phi(w,k) = (w^*(w))^k$, where

$$w^* = \sum_{i=0}^{\infty}(-1)^i w_i 2^i,$$

w_i, being the coefficients in the dyadic expansion of $w \in P$.

It is important to notice that in any case the definition of the function ϕ, and in this way of the Gibbs derivative, relates to the ordering of group characters or unitary irreducible group representations of G.

For example, as noticed above, the Butzer-Wagner dyadic derivative has been defined for the Kaczmarcz and Paley ordered Walsh functions.

From the very beginning of the theory of Gibbs differentiation this was considered as a deficiency of the theory. The problem has been discussed by J.E. Gibbs and several other authors, in particular in details by C.W. Onneweer [41], who endeavored to erase it by suggesting new definitions of derivatives on p-adic and p-series fields.

Another definition of dyadic derivative were offered in [42] and compared with some other definitions of that operator. At the same time, as is noticed in [37], different orderings of group characters or unitary irreducible representations could offer for a given group G the family of Gibbs derivatives some of which could be potentially more convenient than others regarding some concrete applications and numerical calculations. The best ordering of group characters or unitary irreducible representations regarding the efficiency of numerical calculation of Gibbs derivatives on finite groups is determined in [66] using the corresponding results for the implementations of FFT on finite groups. Note that depending on the range of the exponent k the given characterization of Gibbs derivatives extends under appropriate conditions to the fractional Gibbs derivatives, and for $k < 0$ subsumes the concept of Gibbs anti-derivatives. See [11], [55], [63], [65], [76], [77] for some particular examples. The uniqueness of the considered class of differential operators is assured by the requirement that the eigenfunctions of Gibbs derivative are the group characters for Abelian groups and the elements of unitary irreducible representations for finite Abelian groups, i.e.,

$$D^k(\chi_w) = a(w,k)\chi_w, \tag{12.4}$$

for Abelian groups, and

$$D^k R_w^{(i,j)} = a(w,k)R_w^{(i,j)}, \tag{12.5}$$

for non-Abelian groups, where χ_w is the w-th group character of an Abelian, and $R_w^{(ij,)}$ is (i,j)-th element of w-th unitary irreducible representation R_w of a non-Abelian group. Note that the eigenvalues $a(w,k)$ of Gibbs derivatives depend on the ordering of group characters or unitary irreducible group representations in the same way as the function $\phi(w,k)$ depend on that ordering.

If a Gibbs derivative is defined so that $\phi(w,k) = w^k$ in the transform domain, then owing to the orthogonality of Fourier transform, $a(w,k) = w^k$ in the original domain, where w is the index of w-th group character or unitary irreducible representation in the corresponding ordering. For example, w could be the index of w-th Walsh function in Kaczmarcz ordering if the definition of Butzer-Wagner dyadic derivative given in [10] is used, or the index of w-th Walsh function in Paley ordering in the case of Butzer-Wagner dyadic derivative introduced in [11].

Alternative definitions of Gibbs derivatives could yield different eigenvalues. For example, the dyadic derivative derived for $p = 2$ from the Onneweer's definition of dyadic derivative on p-series fields yields different eigenvalues from those of the strong Butzer-Wagner dyadic derivative, since in that case

$$D(wal_k) = 2^n wal_k \quad 2^n \leq k \leq 2^{n+1}, \quad n = 0, 1, \ldots. \tag{12.6}$$

Properties of Gibbs derivatives

Besides linearity, the relation (12.3) and its consequence (12.4) or respectively (12.5), the main properties of Gibbs derivatives could be given as follows.

1 *The derivative of a constant*
$Df = 0 \in K$, iff f is a constant function,

2 *Convolution property*

$$D(f_1 * f_2) = (Df_1) * f_2 = f_1 * (Df_2) \in K(G),$$

where $*$ denotes the convolution in $K(G)$.

3 The group characters χ_w for Abelian groups and the functions $f_{i,j}(z) = R^{(i,j)}(x)$ for finite non-Abelian groups are infinitely many times Gibbs differentiable functions.

4 The Gibbs derivatives do not obey the product rule, i.e., it is false that

$$D(f_1 \cdot f_2) = f_1(Df_2) + (Df_1)f_2, \quad \forall f_1, f_2 \in K(G).$$

Notice that the product rule is used as a base for the introduction of *Ritt-Kolchin derivatives* [35], [52] and, therefore, it follows that the Gibbs derivatives can not be involved in that class of differential operators [15].

5 *Shift invariance*

$$D(T_a f) = T_a(Df), \quad \forall f \in K(G),$$

where T_a denotes the shift operator on G defined as $T_a f(x) = f(x \circ a)$, where $\circ$ denotes the group operation on G.

6 *Haar integral of the derivative*

$$\int_G Df = 0 \in K.$$

7 D is a closed operator in $K(G)$.

We infer by the inspection of many particular Gibbs derivatives that the presented general characterization can be given, but we do not have any pretention to subsume all existing particular cases. In any case, the properties 1-7 can be recognized in the presented or in some slightly modified form in the very most of the particular examples of Gibbs derivatives.

For some generalized product rules for finite dyadic derivative, see [64].

The product rule for extended Butzer-Wagner dyadic derivative valid for Walsh functions is given in [8]. Regarding relationship of Gibbs derivatives with some other differential operators, note that a relationship between strong dyadic derivative and classical *Dini derivatives* were given in [13], [57].

The relationship of Gibbs differentiation with classical Newton-Leibniz differentiation were discussed in [22].

4. Closing Remarks

Trying to estimate and appreciate the role of Gibbs derivatives on the occasion of 40th anniversary of its introduction and 35th and 30th anniversary of two important generalizations by P.L. Butzer and H.J. Wagner, C.W. Onneweer, and other authors, we want to point out the following.

1 Gibbs derivatives enable the transfer of differentiation from the real line to different discrete, and otherwise, not necessarily Abelian structures.

2 Through some particular Gibbs derivatives the class of functions differentiable in some sense, in this case, Gibbs sense, is greatly extended.

3 Some Gibbs derivatives have found interesting applications in different areas as, for example, logic design, statistics, sampling theory, system theory and signal processing, see [26] for the relevant references.

4 Efficiently characterized by Fourier coefficients on groups, Gibbs derivatives can be considered as a part of abstract harmonic analysis, giving to

this mathematical discipline another quality, since relate it with a differential calculus in the same way as the classical Fourier analysis is related to Newton-Leibniz differentiation.

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Chapter 13

WALSH-FOURIER ANALYSIS OF BOOLEAN COMBINERS IN CRYPTOGRAPHY

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Abstract The paper presents a brief overview of applications of Walsh functions in cryptography.

1. Introduction

The theory of Walsh functions goes back to the original paper of Walsh (1923). This was followed by contributions of Paley, Fine and others in pure mathematics. After WWII interest in *communication engineering* and *signal processing* arose and research, mainly in the USA-there at Jet Propulsion Laboratory in Pasadena but also at companies and universities was done. From 1970 on regularly conferences first at the *Naval Research Laboratory* in Washington D.C., were started, with *H. Harmuth* (see, for example, [3], [4]) as its main speaker. It is most likely that at this time the importance of Walsh functions for the characterization of Boolean function as they are applied in cryptography was already known to different organizations but was kept confidential. [1]

2. Walsh Functions: General overview on the theory

Walsh functions have become important for the analysis of Boolean functions by their application in cryptography in combiners and also in S-boxes. Their mathematical theory is highly developed. They are *character functions* of a specific abelian group, *the dyadic group* and the related theory is a special

[1]This paper is a part of a lecture series of the author on Cryptology and is addressed to Non-specialists in the field of Walsh functions. Additional citations may be found in the LNCS publications of the EUROCRYPT and CRYPTO Conferences.

case of the theory characters and of the field of *abstract harmonic analysis* (see for example the book of Rudin [8] or Hewitt-Ross [5]). The case we are dealing with is given by the finite dyadic group $D(n)$ which is the n-fold direct product of the cyclic group Z_2. In this case the theory becomes a part of *linear algebra*.

The *dyadic group* $D(n)$ is defined by $D(n) := (B^n, \oplus)$, its elements $x = (x_0, x_1, \ldots, x_{n-1})$ are Boolean n-tuples, the addition $x \oplus y$ of the elements x and y is coordinate-wise done.

A *Boolean function* f is defined by a function f from B^n to B.

Let $Z(n)$ denote the set $Z(n) := \{0, 1, \ldots, 2^n - 1\}$. There exists a one to one correspondence between $Z(n)$ and $D(n)$ by the function bin with

$$bin(x_0 + 2x_1 + \ldots + 2^{n-1}x_{n-1}) := (x_0, x_1, \ldots, x_{n-1}).$$

For elements t from $Z(n)$ we use often the notation $t = (t_0, t_1, \ldots, t_{n-1})$ and extend the xor operation $\oplus$ also to $Z(n)$.

Walsh functions $w(s, \cdot)$ are usually defined as real-valued functions $w(s, \cdot) : Z(n) \to R$ by

$$w(s,t) = (-1)^{<s,t>}, \quad s \in Z(n),$$

where $< s, t >$ denotes the inner product $s_0t_0 + s_1t_1 + \cdots + s_{n-1}t_{n-1}$ of s with t.

Walsh functions $w(s, \cdot)$ take only values $+1$ and -1. The *Walsh transform* $\hat{F}$ of a function $F : Z(n) \to R$ is defined by

$$\hat{F}(s) := \sum_t F(t)w(s,t).$$

The *inverse Walsh transform* of $\hat{F}$ is given by

$$f(t) = \frac{1}{2^n} \sum_s \hat{F}(s)w(s,t).$$

Let F, G denote functions from $Z(n)$ to R. *The dyadic convolution product* $F * G$ is defined as the function

$$(F * G)(t) = \sum_a F(t \oplus a)G(a).$$

For dyadic convolution the following theorem called *the dyadic convolution theorem* is valid

$$(\widehat{F * G}) = \hat{F} \cdot \hat{G}.$$

Notice that for $F * G = E$ (E the function $E(0) = 1$, $E(t) = 0$, else) it follows $F = E$.

Notice that for $F * G = E$ (E the function $E(0) = 1$, $E(t) = 0$ else) it follows $F = E$.

Let F_a denote the a-dyadic shifted function $F_a(t) := F(t \oplus a)$. The following *dyadic shifting theorem* is easy to prove

$$\hat{F}_a = w(a, \cdot)\hat{F}.$$

From the formula for the Walsh transform of a function F, we get for $\hat{F}(0)$ the value

$$\hat{F}(0) = \sum_t F(t),$$

and from the formula for the inverse Walsh transform

$$F(0) = \frac{1}{2^n} \sum_s \hat{F}(s).$$

The dyadic cross correlation function $DCC(F, G)$ of functions F and G is defined by

$$DCC(F, G) = \sum_t F(t \oplus a)G(t).$$

The dyadic autocorrelation function $DAC(F)$ of a function F is defined by

$$DAC(F)(a) = \sum_t F(t \oplus a)F(t).$$

For the DAC of a function F the following theorem can be proven

$$\widehat{DAC}(F) = F^2,$$

which is a mathematical expression of *the Theorem of Wiener-Khintchin.*

Of specific interest are functions F which take (as the Walsh functions) only values $+1$ and -1 on $Z(n)$. The following theorem characterizes such functions by spectral properties.

THEOREM 13.1 *A function F is a "$+1/-1$ function" if and only if the following equation is valid*

$$\hat{F} * \hat{F} = 2^n E.$$

An interesting theorem is the following.

THEOREM 13.2 *(Theorem of Liedl)*
Let F be a polynomial of degree $m < n$. Then, $\hat{F}(s) = 0$ for all s with $\|s\|_H > m$, where $\|s\|_H$ denotes the Hamming weight *of s.*

3. Walsh Fourier Analysis of Boolean Functions

Applications of the theory of Walsh functions in the field of *cryptology* deal mainly with Boolean functions $f : B^n \to B$. Any such function f has a corresponding $+1/-1$ function F which is given by $F(t) = (-1)^{f(x)}$ where $x = bin(t)$. In the following, F has always this meaning.

The Walsh transform $\hat{f}$ of a Boolean function f is defined in the following way

$$\hat{f}(y) = \sum_x (-1)^{f(x)} (-1)^{<x,y>},$$

or since $f(x) + < y, x >$ (mod 2) $= f(x) \oplus < y, x >$ we have also

$$\hat{f}(y) = \sum_x (-1)^{f(x) \oplus <x,y>}.$$

It is to observe that the Walsh transform $\hat{f}$ of a Boolean function f is real-valued.

The dyadic autocorrelation and the dyadic cross correlation of a Boolean function f is defined by the DAC and DCC of the associated $+1/-1$ function F:

$$DAC(f) := DAC(F)$$

and

$$DCC(f) := DCC(F).$$

The following results for Boolean functions f are valid:

$$DAC(f)(0) = 2^n,$$

and

$$\|f \oplus f_a\|_H = 1/2 - 1/2^{n+1} DAC(f)(a).$$

It can be observed that for $a = 0$ as expected $\|f \oplus f\|_H = 0$. Furthermore that the Hamming distance of f and f_a for $a \neq 0$ is close to $\frac{1}{2}$ if $DAC(f)(a)$ is small. This is the case for the Boolean functions $f : D(2^n - 1) \to B$ which are generated by a maximum length linear feedback shift register MLFSR of length n (pseudo random code-words).

A main application of the Walsh transform in cryptology is given by the spectral characterization of Boolean functions.

A Boolean function f on $D(n)$ is called *balanced* if

$$card\{x : f(x) = 0\} = card\{x : f(x) = 1\} = 2^n/2.$$

We have the "theorem": A function f is balanced if $\hat{f}(0) = 2^n/2$.

A Boolean function f satisfies by definition *the propagation criteria with respect to* $a \in D(n)$ if $f \oplus f_a$ is balanced. Here f_a denotes the dyadic a-shift of f which is given by $f_a(x) := f(x \oplus a)$.

A Boolean function f satisfies by definition the propagation criteria of degree k if it satisfies the propagation criteria for all $a \in D(n)$ with $0 < \|a\|_H = k$. In the case $k = 1$ we say that f satisfies *the Strict Avalanche Criteria* (SAC). The following theorem can be proven for the SAC of a Boolean function f:

THEOREM 13.3 *([1])*
A Boolean function f satisfies the Strict Avalanche Criteria if

$$\sum_s (\hat{f})^2(s)(-1)^{s_i} = 0,$$

for all i with $1 \leq i \leq n$.

The distance $d(f, g)$ between two Boolean functions f and g is given by

$$d(f, g) = \|f \oplus g\|_H.$$

Linear Boolean functions $l(y)$ are of the form $l(y)(x) =< y, x >$ or $l(y) = 1 \oplus < y, x >$.

A degree of non-linearity of a Boolean function f can be measured by its distance to a linear Boolean function. The following theorem allows to express the distance of a Boolean function f to the linear functions $l(y)$ by means of its spectrum:

THEOREM 13.4 *For a Boolean function f and a linear Boolean function $l(y)$*

$$d(f, l(y)) = \frac{1}{2}(2^n \hat{f}(y)).$$

In stream cipher architectures the analysis of the Boolean function which realizes a static combiner is of specific importance. To block correlation attacks to investigate the used secret key a sufficient degree m of correlation immunity of such a function is required. In this respect the following definition is introduced:

A Boolean function f is called to be *correlation immune* of order m if $f(x_1, x_2, \ldots, x_n)$ is statistically independent from every k-tupel, where $k < m + 1$, when considered as independent uniformly distributed binary random variables of stochastic processes $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$.

For the characterization of a Boolean function with respect to its correlation immunity the following theorem is of importance.

THEOREM 13.5 *([10])*
A Boolean function f is correlation immune of order m if and only if $\hat{f}(y) = 0$ for all y with $\|y\|_H \leq m$, where $\|y\|_H$ denotes the Hamming weight of y.

In the terminology of Walsh-Fourier analysis this means, that a Boolean function f which is correlation immune of order m contains in its Walsh-Fourier representation only Walsh functions, which are a product of more than m *Rademacher functions*. The related Walsh-Fourier spectrum $\hat{f}$ can therefore be considered as a nonlinear function which compares to polynomials of higher degree than m.

4. Design of Boolean Function Combiners

The determination of a Boolean function which meets the necessary requirements is an important mathematical task in the cryptography of stream ciphers. We explore in detail the following properties, which have some relevance.

If $x_1, x_2, x_3, \ldots, x_n$ denotes the pseudo random streams received by the combiner C, then the resulting output stream y of the considered combiner $C(x_1, x_2, x_3, \ldots, x_n)$ should be "cryptologically improved" compared to the individual input streams x_i ($i = 1, 2, 3, \ldots, n$).

A combiner C must not "leak" (should have a strong one way property to make cryptanaysis difficult).

The design of combiners for strong pseudo random generators used in cryptography is usually a part of a trade secret of companies. However there are a number of published results which can give an orientation. Most of publications deal with static combiners, based on Boolean functions, only a few results are known for dynamic combiner.

A Boolean combiner can be realized by a properly chosen Boolean function C from B^n to $B := \{0, 1\}$. A Boolean combiner can be represented either by a table or by a Boolean expression. Usually it is to assume that C is given by its *Algebraic Normal Form* $ANF(C)$.

One of the most important requirements in the design of Boolean combiners concerns the degree of correlation immunity $I(C)$ to avoid leaking with respect to the correlation attack (described by Siegenthaler [9] and Golic [2]) $I(C)$ can be determined by spectral properties of the Walsh-Fourier transform $WFT(C)$ of C. A sufficient degree $I(C)$ needs a certain degree of nonlinearity of the discrete polynomial associated to C. The results, which are already described in Section 2 were derived by the work of Xiao and Massey [10].

It is possible to construct a sufficiently large number of correlation immune Boolean functions for any desired degree m [6]. Other results which are derived by Siegenthaler [9] are based on repeated algebraic computations.

If the required degree $I(C)$ of correlation immunity of a combiner C is given, then the following recipe for the construction of a combiner C with degree $I(C) = m$ can be applied:

1 Define C by $C(x_1, x_2, \ldots, x_n) := x_1 \oplus x_2 \oplus \cdots x_m \oplus g(x_{m+1}, \ldots, x_n)$ with a Boolean function $g : B^{n-m} \to B$.

2 Chose g such that the additional required properties of C are fulfilled.

In the following we explore some additional features and their spectral representation by Walsh-Fourier representations which are used in combiner design. In doing this we have to distinguish between *static combiners* represented by Boolean functions (Boolean function combiner) and *dynamic combiners* (FSM combiners, also called in cryptography "combiners with memory") represented by *finite state machines*.

In both cases it is the goal to derive from observed output bits

$$y(0)y(1)y(2), \ldots, y(k), \ldots$$

of the combiner C some knowledge about the input streams $x_1, x_2, \ldots, x_n$ and the "machines" (specifically their initial states) $M_1, M_2, \ldots, M_n$ which generate it. To get such a useful knowledge for the mounting of an attack this should be computational hard. In the case of dynamic combiners our consideration is restricted to finite state machines which are finite memory machines. In this case the problem of the design of a dynamic combiner can be reduced to the design of a Boolean function combiner.

Boolean function combiners are designed by switching functions f such that

1 The solution of the system of equations $f(x(i)) = y(i), i = 0, 1, 2, \ldots, k$; $x(i) = (x_1(i), x_2(i), \ldots, x_n(i))$ is computational hard.

2 A correlation analysis between the output stream y and the individual input streams x_i $(i = 1, 2, \ldots, n)$ shows no results regardless of the length of the applied streams y and x_i.

The condition (1) requires the highly nonlinear functions f. In the condition (2), in contradiction, however, certain linear component of f is required to meet correlation immunity requirements.

Different ways to represent switching functions f are known:

1 By *the disjunctive form DF*,

2 By *the conjunctive form CF*,

3 By *the algebraic normal form ANF* (a *multivariate polynomial*).

In cryptanalysis it is for algebraic reasons often desirable to use the ANF of a Boolean function f which is given by

$$\begin{aligned} ANF(f) \quad := \quad & a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_{1,2}x_1x_2 + a_{1,3}x_1x_3 \\ & + \cdots + a_{n-1,n}x_{n-1}x_n + a_{1,2,3}x_1x_2x_3 + a_{1,2,4}x_1x_2x_4 \\ & + \cdots + a_{1,2,3,4,\ldots,n}x_1x_2\cdots x_n, \end{aligned}$$

there exist methods to compute DF, CF and ANF from any each other.

In Cryptology the following criteria are considered as useful in the design and analysis of Boolean combiners and Boolean networks ("S-boxes")

1 Balance,

2 Nonlinear order,

3 Correlation immunity,

4 Bentness,

5 Distance to linear structures,

6 Strict avalanche criterion,

7 Propagation characteristic,

8 Global avalanche criterion.

The criteria (1)-(8) can be defined as shown in the following. Also their characterization, if possible, in the spectral domain is given:

A Boolean function f is called *balanced* if

$$card\{x : f(x) = 1\} = card\{x : f(x) = 0\}.$$

The nonlinear order of f is defined by the maximal numbers of variables which appear in the $ANF(f)$.

A Boolean function f is *correlation-immune of order* m if the value of f is statistically independent from any m-tupel (compare with a more detailed definition in Section 2 of this paper).

These properties can be characterized by the Walsh-Fourier spectrum $\hat{f}$ of f in the following way:

THEOREM 13.6 *A Boolean function f is balanced iff $\hat{f}(0) = 0$.*

THEOREM 13.7 *A Boolean function f is correlation immune of order m iff $\hat{f}(w) = 0$ for all w with the Hamming weight $\|w\|_H \leq m$.*

THEOREM 13.8 *A Boolean function f is said to satisfy the strict avalanche criterion (SAC) if*

$$Pr\{f(x) \oplus f(x \oplus a) = 1\} = \frac{1}{2}$$

for $\|a\| = 1$.

THEOREM 13.9 *A Boolean function f satisfies the propagation characteristic (PC) of degree k if*

$$Pr\{f(x) \oplus f(x \oplus a) = 1\} = \frac{1}{2},$$

for $1 \leq \|a\| \leq k$.

REMARK 13.1 *The perfect nonlinearity requires a PC of degree n.*

THEOREM 13.10 *The global avalanche criterion GAC of a Boolean function f can be characterized by the dyadic autocorrelation function $DAC(f)$ of f which is given by*

$$DAC(f) := \sum_x F(x)F(x \oplus a).$$

A "good" GAC means that $DAC(f)$ is close to zero for almost all nonzero values of a and for $a = 0$ we should have $DAC(f)(0) = 2^n$. The Walsh-Fourier transform $WFT(DAC(f))$ is according to the Wiener-Khintchin theorem *the Walsh Power Spectrum* $P(f)$ of the Boolean function f. For functions f with good GAC the related $P(f)$ is almost constant (has a "white noise" characteristic).

Bent functions are Boolean functions f which satisfy the propagation characteristic PC by degree n. For bent functions the following theorem is valid.

THEOREM 13.11 *A Boolean function f is a bent function if the modulus of $\hat{f}$ ($\hat{f}$ the Walsh transform of f) is constant with $\hat{f}(w) = 2^n/2$ for all $w \in GF(2)^n$.*

Satisfying the criteria (1)-(8) may lead to conflicts. Such examples are as follows:

1 Usually it is required that C is balanced, so it cannot be a bent function.

2 Bent functions does not exist if n is odd.

3 High linear order means low degree of correlation immunity.

To avoid a trade-off of the kind (1) it can be suggested to use static combiners C of the form

$$C = x_1 \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_m \oplus C'(x_{m+1}, x_{m+2}, \cdots, x_n),$$

which is of the correlation immunity of degree m.

To meet additional criteria the Boolean function C' has to be designed accordingly.

5. Finite Memory FSM Combiners

For special cases of finite state machines the design of dynamic combiners can be reduced to the design of a static (Boolean) combiner. One class is given by finite state machines which possess the "finite memory property". Such finite state machines are called *Finite Memory Machines* (FMM). This leads to the concept of a *Finite Memory FSM combiner* (FMM combiner).

In the following the main results for the design of FMM-combiners are stated, see [7].

DEFINITION 13.1 *A finite state machine FSM has a finite memory of degree i if any simple experiment (w, v) observed on the FSM determines uniquely the reached state q as soon as the length of the experiment is equal to i or is larger than i, length $(w, v) = i$ or length $(w, v) > i$.*

There exist efficient algorithms to determine if a given FSM is a finite memory machine FMM and also to determine the degree i. If a FSM is a FMM then there is the possibility to compute the associated canonical shift register representation, which contains two feed-forward registers for the input and the output together with the Boolean output function $f(FMM)$.

In a FMM state transition is reduced to simple shift operation.

The following procedure for constructing a FMM combiner is suggested:

1 Feed the input streams $x_1, x_2, \ldots, x_n$ to the associated feed-forward shift register $R_1, R_2, \ldots, R_n$ of lengths $m_1, m_2, \ldots, m_n$.

2 Feed the output stream y to a feed-forward shift register R of length m.

3 Take all register states as inputs of the output function $f(FMM)$.

The resulting FMM has a finite memory of degree

$$i = max(m_1, m_2, \ldots, m_n, m).$$

The following steps can use the methods for the cryptographic design of the Boolean function $f(FMM)$ with the cryptanalytic methods known for Boolean function combiners. To camouflage the design the change of the state coordinates of the FMM is recommended to change by state assignment the states such that the FMM canonical form disappears so that, after implementation, it is not immediately recognized as a FMM combiner on hardware blueprints.

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Chapter 14

WALSH SERIES OF COUNTABLY MANY VARIABLES

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Abstract This paper presents some results on the convergence of the Walsh-Fourier series for the functions of countably many variables.

1. Introduction and Background Work

Let $\mathbf{T}^\infty$ and $[0,1]^\infty$ are the Cartesian product of countably many of one-dimensional tori $\mathbf{T} = \mathbf{R}/\mathbf{Z}$:

$$\mathbf{T}^\infty = \{x = (x_1, \ldots, x_n, \ldots) : 0 \leq x_n < 1, n \in \mathbf{N}\}$$

and the Cartesian product of countably many segments $[0,1]$, respectively.

The torus $\mathbf{T}^\infty$ and the cube $[0,1]^\infty$ are equipped with the *Tychonoff topology*.

The Lebesque measure is defined on $\mathbf{T}^\infty$ and on $[0,1]^\infty$ as on the product of the spaces with one-dimensional Lebesque measure. Hence it is possible to consider L^p and other spaces on $\mathbf{T}^\infty$ and on $[0,1]^\infty$.

H. Steinhaus (1930) translated a probability result of A.N. Kolmogorov into terms of the theory of functions:

A trigonometric series of the form $\sum\limits_{k=1}^{\infty} a_k e^{2\pi i x_k}$ *converges (diverges) almost everywhere on* $\mathbf{T}^\infty$ *if* $\sum\limits_{k=1}^{\infty} |a_k|^2 < \infty$ *(*$\sum\limits_{k=1}^{\infty} |a_k|^2 = \infty$*).*

The system of functions

$$\theta_{n_1,\ldots,n_p}(x) = \prod_{r=1}^{p} e^{2\pi i n_r x_r}, p \in \mathbf{N}, n_r \in \mathbf{Z}, x = (x_1, \ldots, x_p, \ldots) \in \mathbf{T}^\infty$$

is called the *Jessen system*.

The Jessen system is a complete orthonormal system on $\mathbf{T}^\infty$.

B. Jessen (1934) [1] obtained the representation of an integrable function as a limit of integrals.

Let us denote by $\mathbf{T}^n$ the n-dimensional torus:

$$\mathbf{T}^n = \{(x_1, \dots, x_n) : 0 \le x_k < 1,\ k = 1, 2, \dots, n\},$$

and by $\mathbf{T}^{n,\infty}$ the infinite dimensional torus:

$$\mathbf{T}^{n,\infty} = \{(x_{n+1}, \dots, x_{n+p}, \dots) : 0 \le x_k < 1, \quad k = n+1, \dots, n+p, \dots\}.$$

Let us denote by $[0,1]^{n,\infty}$ the infinite dimensional cube

$$[0,1]^{n,\infty} = \{(x_{n+1}, \dots, x_{n+p}, \dots) \colon 0 \le x_k \le 1, \quad k = n+1, \dots n+p, \dots\},$$

$[0,1]^n$ — n-dimensional cube.

B. Jessen [1] proved the following

THEOREM 14.1 *Let $f \in L(\mathbf{T}^\infty)$. Then the sequence of functions*

$$f_n(x) = \int\limits_{\mathbf{T}^{n,\infty}} f(x) dx_{n+1} \dots dx_{n+p} \dots, \ n \in \mathbf{N}.$$

converges a.e. on $\mathbf{T}^\infty$ to f, that is

$$\lim_{n\to\infty} \int\limits_{\mathbf{T}^{n,\infty}} f(x) dx_{n+1} \dots dx_{n+p} \dots = f(x), \quad a.e.\ x \in \mathbf{T}^\infty.$$

If $f \in L^p(\mathbf{T}^\infty), p \ge 1$, then $f_n \in L^p(\mathbf{T}^\infty)$ and sequence $\{f_n\}$ converges to f in $L^p(\mathbf{T}^\infty)$.

Of course, Jessen's theorem is correct for functions $f \in L([0,1]^\infty)$ and

$$f_n(x) = \int_{[0,1]^{n,\infty}} f(x)\, dx_{n+1} \dots dx_{n+p} \dots, \quad n \in \mathbf{N}. \tag{14.1}$$

This Jessen's theorem allows to solve some questions on Fourier series of functions on $\mathbf{T}^\infty$ and $[0,1]^\infty$.

Let us remark that T.I. Ahobadze (1986) considered systems of functions on $\mathbf{T}^\infty$ and in particular the Jessen system.

2. Series in Terms of Infinite Dimensional Walsh System

We will consider Walsh system on $[0,1]^\infty$:

$$W_{n_1,\dots,n_p}(x) = \prod_{r=1}^{p} w_{n_r}(x_r), p \in \mathbf{N}, n_r \in \mathbf{Z}_+, x = (x_1, \dots, x_p, \dots) \in [0,1]^\infty,$$

where $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$ and $w_{n_r}(x_r)$ are one-dimensional Walsh functions.

This system is a complete orthonormal system on $[0,1]^\infty$.

F. Schipp [2] considered generalized product systems and obtained results on L^p - norm convergence ($1 < p < \infty$) of Fourier expansions with respect to the product system of independent systems and with respect to Vilenkin systems.

Let ${\mathbf{Z}_+}^{<\infty}$ be the set of infinite dimensional vectors $n = (n_1, \ldots, n_p, \ldots)$ with $n_p \in \mathbf{Z}_+$ and only a finite number of n_p are different from zero.

We will consider series on infinite dimensional Walsh system

$$\sum_{n \in {\mathbf{Z}_+}^{<\infty}} a_n W_n(x), \qquad \text{where } x \in [0,1]^\infty,\ a_n \in \mathbf{R}. \tag{14.2}$$

If for $n = (n_1, \ldots, n_p, \ldots) \in {\mathbf{Z}_+}^{<\infty}$ the coordinates $n_k = 0$ for $k > p$, then we'll use the following notations for a_n: $a_n = a_{n_1,\ldots,n_p}$. Each of these notation determines n and a_n uniquely.

Denote the rectangular partial sums by

$$S_{p,N_1,\ldots,N_p}(x) = \sum_{n_1=0}^{N_1} \ldots \sum_{n_p=0}^{N_p} a_{n_1,\ldots,n_p} w_{n_1}(x_1) \ldots w_{n_p}(x_p), \tag{14.3}$$

where $p \in \mathbf{N}, N_1, \ldots, N_p \in \mathbf{Z}_+$, $x \in [0,1]^\infty$. If $N_1 = \ldots = N_p$ these are cubic partial sums.

We say that the series (14.2) converges rectangularly at the point $x \in [0,1]^\infty$ to the number s if for any $\varepsilon > 0$ there exists an index P such that for any $p \geq P$ we can find a natural number N such that for any $N_1, \ldots, N_p \geq N$ the inequality $|S_{p,N_1,\ldots,N_p}(x) - s| < \varepsilon$ is fulfilled.

The series (14.2) is called convergent over cubes if we consider only cubic partial sums in this definition.

We say that the series (14.2) converges rectangularly (over cubes) in strengthened sense at the point $x \in [0,1]^\infty$ to the number s if for any p multiple Walsh series $\sum\limits_{n_1,\ldots n_p} a_{n_1,\ldots,n_p} w_{n_1}(x_1) \ldots w_{n_p}(x_p)$ converges rectangularly (over cubes) at the point x and there exist limit

$$\lim_{p \to \infty} \sum_{n_1,\ldots,n_p} a_{n_1,\ldots,n_p} w_{n_1}(x_1) \ldots w_{n_p}(x_p) = s.$$

It is easy to see if the series (14.2) converges in the strengthened sense, then it converges.

F. Móricz [3] proved that if $f \in L^2([0,1]^2)$, then the square partial sums of the Walsh-Fourier series of f converge to f almost everywhere on $[0,1]^2$. This result is correctly and in d-dimensional case. It follows from this and from Jessen's theorem A

THEOREM 14.2 *If $f \in L^2([0,1]^\infty)$, then the Fourier series of f converges over cubes in strengthened sense to f a.e. on $[0,1]^\infty$.*

Let $\varphi\colon [0,+\infty) \to [0,+\infty)$ be an increasing function. Then we denote by $\varphi(L)([0,1]^\infty)$ the set of all measurable functions f on $[0,1]^\infty$ such that $\int\limits_{[0,1]^\infty} \varphi(|f(x)|)dx < \infty$. Similarly, $\varphi(L)([0,1]^n)$ is the set of all measurable functions f on $[0,1]^n$, such that $\int\limits_{[0,1]^n} \varphi(|f(x)|)\,dx_1 \dots dx_n < \infty$.

LEMMA 14.1 *Let $\varphi\colon [0,+\infty) \to [0,+\infty)$ be an increasing convex function. If $f \in \varphi(L)([0,1]^\infty)$, then f_n ($n \in \mathbf{N}$) (see (14.1)) are from $\varphi(L)([0,1]^n)$. If we consider f_n as functions on $[0,1]^\infty$, then $f_n \in \varphi(L)([0,1]^\infty)$, $n \in \mathbf{N}$.*

Proof. Recall the Jensen inequality in general form. Let μ be probability measure on measurable space $(X,\mathcal{A})$, g — μ-integrable function with values in the domain of definition of convex function ψ and $\psi(f)$ is integrable function. Then,

$$\psi\Big(\int_X g(x)\mu(dx)\Big) \le \int_X \psi(f(x))\mu(dx).$$

We have in our case $X = [0,1]^\infty$, μ — Lebesque measure, $g = |f_n|$, $\varphi = \psi$. It follows from increasing of function φ and from Jensen's inequality that

$$\begin{aligned} \varphi(|f_n(x)|) &\le \varphi\Big(\int_{[0,1]^{n,\infty}} |f(x)|\,dx_{n+1}\dots dx_{n+p}\dots\Big) \\ &\le \int_{[0,1]^{n,\infty}} \varphi(|f(x)|)\,dx_{n+1}\dots dx_{n+p}\dots . \end{aligned}$$

Now we obtain by integrating this inequality on $[0,1]^n$

$$\int_{[0,1]^n} \varphi(|f_n(x)|)\,dx_1\dots dx_n \le \int_{[0,1]^\infty} \varphi(|f(x)|)\,dx < \infty.$$

Lemma is proven.

It is known and follows from the Jessen-Marcinkiewich-Zygmund theorem on the strong differentiation of integrals [4], Ch. XVII, paragraph 2, and from the form of partial sums of multiple Fourier-Haar series that if $f \in L(\log^+ L)^{d-1}([0,1]^d)$, then rectangular partial sums of the Haar-Fourier series of f converge to f almost everywhere on $[0,1]^d$.

For Walsh-Fourier series we obtain by this and by Lemma and Theorem A the following

THEOREM 14.3 *If* $f \in \bigcap_{d=1}^{\infty} L(\log^+ L)^d(\mathbf{T}^\infty)$, *then the rectangular partial sums* $S_{2^{n_1},2^{n_2},\ldots,2^{n_p}}$ *of the Walsh-Fourier series of* f *converges in strengthened sense to* f *a.e. on* $[0,1]^\infty$.

REMARK 14.1 *The condition* $f \in \bigcap_{d=1}^{\infty} L(\log^+ L)^d([0,1]^\infty)$ *may be given in the equivalent form: there exist an increasing function* $h(x)$ *defined on* $[0,\infty)$, $h(x) \geq 0$, $\lim_{x\to\infty} h(x) = +\infty$ *such that*

$$\int\limits_{[0,1]^\infty} |f(x)|(\log^+ |f(x)|)^{h(|f(x)|)}dx < \infty.$$

REMARK 14.2 *It follows from S. Saks [4], Ch. XVII, paragraph 2, and T.S. Zerekidze [5] results that any class* $L(\log^+ L)^d([0,1]^\infty)$ *contains a function for which the Theorem 14.3 is not correct.*

REMARK 14.3 *It follows from results by D.K. Sanadze and Sh.V. Kheladze [6] that under conditions of Theorem 14.3 the rectangular partial sums of the Walsh-Fourier series of* f *converge to* f *a.e. on* $[0,1]^\infty$ *if all but one indices are lacunary.*

Let us compare the Theorem 14.3 with the result for the Jessen system

THEOREM 14.4 *If* $f \in \bigcap_{d=1}^{\infty} L(\log^+ L)^d(\mathbf{T}^\infty)$, *then the trigonometric Fourier series of* f *converges over cubes in the strengthened sense to* f *a.e. on* $\mathbf{T}^\infty$.

(The proof of Theorem 14.4 can be found in [7].)

REMARK 14.4 *It follows from Konyagin's result [8] that any class* $L(\log^+ L)^d(\mathbf{T}^\infty)$ *contains a function with the Fourier series divergent over cubes everywhere on* $\mathbf{T}^\infty$.

F. Weisz proved (see in [9] the 2-dimensional case) that if $f \in L[0,1]^n$ and

$$S_{k,\ldots,k}(f,x) = \sum_{m_1=0}^{k-1} \cdots \sum_{m_1=0}^{k-1} a_{m_1,\ldots,m_n} w_{m_1}(x_1)\ldots w_{m_n}(x_n)$$

are cubic partial sums of Fourier series of f ($k = 1,2,3,\ldots$), then

$$\lim_{k\to\infty} \frac{S_{1,\ldots,1}(f,x) + S_{2,\ldots,2}(f,x) + \ldots + S_{k,\ldots,k}(f,x)}{k} = f(x) \quad \text{a.e. on } [0,1]^n.$$

From this and from Theorem 14.1 we obtain:
If $f \in L([0,1]^\infty)$ *and* $S_{n,k,\ldots,k}(f,x)$ *cubic partial sums of* f *(see (3)) then*

$$\lim_{n\to\infty} \lim_{k\to\infty} \frac{S_{n,1,\ldots,1}(f,x) + S_{n,2,\ldots,2}(f,x) + \ldots + S_{n,k,\ldots,k}(f,x)}{k} = f(x)$$

a.e. on $[0,1]^\infty$.

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Chapter 15

SETS OF UNIQUENESS FOR MULTIPLE WALSH SERIES

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Abstract This brief review is devoted to the uniqueness problem for multiple Walsh series.

1. Introduction

The theory of uniqueness for orthogonal system originated from the well-known Cantor's theorem [1872]:

If a trigonometric series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ converges to zero everywhere on $[0, 2\pi]$, then this series is identically zero, i.e., $a_n = 0$ for all $n \in \mathbf{Z}$.

We remind the principal definitions.

Let $\{\psi_n\}$ be an orthogonal system of functions on some set A. Consider a series

$$\sum_n c_n \psi_n. \tag{15.1}$$

A set $L \subset A$ is called a *set of uniqueness* (or in short: a *U-set*) for the system $\{\psi_n\}$ if from the convergence of the series (15.1) to zero outside the set L it follows that $c_n = 0$ for all n. If a set L is not a U-set for the system $\{\psi_n\}$ then it is called a *set of multiplicity* (or in short: a *M-set*) for the system $\{\psi_n\}$. With this terminology the Cantor theorem states that $\emptyset$ is a U-set for trigonometric series.

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2. Walsh-Paley System

Now we remind some basic results for one-dimensional Walsh-Paley system.

Vilenkin [13] first proved that $\emptyset$ is a U-set (for Walsh series). Šneider [12], and, independently, Fine [3] have shown that every finite or countable set $L \subset [0,1]$ is a U-set. Šneider [12] also proved that every set $L \subset [0,1]$ of positive Lebesgue measure is a M-set and that there are uncountable Borel U-sets. Coury [2] has constructed a Borel M-set whose Lebesgue measure is zero. Skvortsov [11] strengthened the last result. He proved that there is a perfect M-set, whose Hausdorff p-measure is zero for all $p > 0$ (the definition of the Hausdorff p-measure is given below). Wade [14] proved that if $L_1,\ L_2, ...$ is a sequence of closed U-sets then $\bigcup_{n=1}^{\infty} L_n$ is also a U-set.

3. Multidimensional Case

Now we consider the multidimensional case. We write G for the *dyadic group*.

Let $\{\omega_n(t)\}_{n=0}^{\infty}$ be the *Walsh-Paley system* on $[0,1]$ or on G. Fix natural $d \geq 1$. Consider the *d-multiple Walsh system* on the G^d, i.e.,

$$\{\omega_{\mathbf{n}}(\mathbf{t})\} = \{\omega_{n_1}(t^1)\cdot\ldots\cdot\omega_{n_d}(t^d)\}, \quad \mathbf{n} = (n_1,\ldots,n_d), \quad \mathbf{t} = (t^1,\ldots,t^d). \tag{15.2}$$

The *d-multiple Walsh series* is defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}}\omega_{\mathbf{n}}(\mathbf{t}) = \sum_{n_1=0}^{\infty}\ldots\sum_{n_d=0}^{\infty} b_{n_1,\ldots,n_d}\prod_{i=1}^{d}\omega_{n_i}(t^i), \tag{15.3}$$

where $b_{\mathbf{n}}$ are real numbers. If $\mathbf{N} = (N_1,\ldots,N_d)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of the series (15.3) at a point $\mathbf{t}$ is

$$S_{\mathbf{N}}(\mathbf{t}) = \sum_{n_1=0}^{N_1-1}\ldots\sum_{n_d=0}^{N_d-1} b_{\mathbf{n}}\omega_{\mathbf{n}}(\mathbf{t}).$$

Let $d \geq 2$; then the series (15.3) *converges rectangularly* to a sum $S(\mathbf{t})$ at a point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \ \text{ as } \ \min_i\{N_i\} \to \infty.$$

Let $\rho \in (0,1]$; then the series (15.3) *converges ρ-regularly* to a sum $S(\mathbf{t})$ at a point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \ \text{ as } \ \min_i\{N_i\} \to \infty \ \text{ and } \ \min_{i,j}\{N_i/N_j\} \geq \rho.$$

The 1-regular convergence we call *cubical*.

It is obvious that if the series (15.3) converges rectangularly to a sum $S(\mathbf{t})$ at a point $\mathbf{t}$ then for every $\rho \in (0,1]$ this series converges ρ-regularly to $S(\mathbf{t})$ at $\mathbf{t}$.

Let $U_{\text{rect},d}$ ($U_{\rho,d}$) denote the class of all U-sets for the system (15.2) with respect to the rectangular (ρ-regular) convergence. Obviously, $U_{\rho,d} \subset U_{\text{rect},d}$ for all $\rho \in (0,1]$.

It follows from results by Skvortsov [11] and Movsisyan [8] that if $L \subset [0,1]^d$ be a countable set, then $L \in U_{\text{rect},d}$. The wide class of $U_{\text{rect},d}$-sets was constructed by Lukomskiĭ [1989]. In this work it was also proved that if $E \subset [0,1]^{d-1}$, then $E \in U_{\text{rect},d-1}$ if and only if $E \times G \in U_{\text{rect},d}$. In particular there exist uncountable Borel sets $L \in U_{\text{rect},d}$. Kholshchevnikova [4] proved the following multidimensional analogue of the Wade result: Let $L_1, L_2, ...$ be a sequence of closed $U_{\text{rect},d}$-sets such that $L_n \subset (0,1)^d$ and mes$L_n = 0$. Then $\bigcup_{n=1}^{\infty} L_n$ is a $U_{\text{rect},d}$-set.

Below the series (15.3) are considered on G^d instead of on $[0,1]^d$. Lukomskiĭ [1996] proved that $\emptyset \in U_{\rho,d}$ for all $\rho \in (0,1]$. In Lukomskiĭ [6] it was proven that $U_{\rho,d} \neq U_{\text{rect},d}$ for all $\rho \in (0,1]$.

We have proved the following results (see Plotnikov [9] and [10]).

THEOREM 15.1 *Let $L \subset G^d$ be a finite set. Then, $L \in U_{1,d}$.*

THEOREM 15.2 *There exist countable sets $L \in U_{1,d}$.*

THEOREM 15.3 *Let $L \subset G^d$ be a countable set. Then, $L \in U_{1/2,d}$.*

Next we use the concepts of the Hausdorff p-measure and the Hausdorff dimension. Let $A \subset \mathbf{R}^d$. *A covering* of a set A is a collection $I = \{\Delta\}$ of sets $\Delta \subset \mathbf{R}^d$ such that $A \subset \bigcup_{\Delta \in I} \Delta$. Then *the Hausdorff p-measure* of a set A is defined by

$$\text{mes}_p A = \liminf_{\varepsilon \to +0} \sum (\text{diam}\, \Delta)^p, \tag{15.4}$$

where $\{\Delta\}$ is a finite or countable covering of a set A by cubes or balls Δ with $\text{diam}\, \Delta < \varepsilon$.

By *the Hausdorff dimension* of a set $A \subset \mathbf{R}^d$ we mean the number $\dim_H A$ equal to

$$\sup\{p \in \mathbf{R},\ \text{mes}_p A > 0\}. \tag{15.5}$$

Define the Hausdorff dimension of a set $L \subset G^d$. Recall that the dyadic group G is a set of the infinite sequences $t = \{t_k\}$ where $t_k = 0$ or 1. Consider the map $\varphi : G \to [0,1]$ given by the formula

$$\varphi(t) = \sum_{k=0}^{\infty} t_k 2^{-k-1}.$$

Let $\phi(\mathbf{t}) = (\varphi(t^1), \ldots, \varphi(t^d))$ for $\mathbf{t} = (t^1, \ldots, t^d)$. Using (15.4) and (15.5) we define the Hausdorff dimension of a set $L \subset G^d$ by

$$\dim_H L = \dim_H(\phi(L)), \quad (\phi(L) \subset \mathbf{R}^d).$$

THEOREM 15.4 *For every $d = 1, 2, \ldots$ there is a perfect set $L \in U_{1,d}$ such that $\dim_H L = d$.*

In the case $d = 1$ the Theorem 15.4 and the results by Skvortsov [1977] are the complement of one another. These theorems show that the property of a set to be a U-set or a M-set does not depend on the Hausdorff dimension of the set.

4. Open Questions

In conclusion we state four open questions which seem to be of interest.

Question 1. *Let $\rho \in (0, 1]$ be chosen. Consider the series (15.3) on $[0, 1]^d$. Is $\emptyset \in U_{\rho,d}$? The similar problem is also open for the multiple trigonometric system.*

Question 2. *Is any set $L \subset [0, 1]^d$ (resp. $L \subset G^d$) with positive Lebesgue measure (resp. Haar measure) a set of multiplicity of d-multiple Walsh system for the rectangular or ρ-regular convergence? The analogous problem is open for the multiple trigonometric system.*

Question 3. *Is any countable set $L \subset G^d$ a set of uniqueness for d-multiple Walsh system for the cubical convergence?*

Question 4. *Consider the series (15.3) on G^d. Are there $0 < \rho_1 < \rho_2 \leq 1$ such that $U_{\rho_1,d} \neq U_{\rho_2,d}$? The similar problem is open for the multiple trigonometric system.*

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Chapter 16

CONSTRUCTION AND PROPERTIES OF DISCRETE WALSH TRANSFORM MATRICES

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Abstract This paper discusses properties properties the discrete Walsh transform for different orderings of Walsh functions. Presented are some methods for construction of the related transform matrices.

1. Introduction

In technical applications [1] *Discrete Walsh Transform* (DWT) is used in three enumerations: Paley, Walsh and Hadamard. In technical literature the Discrete Walsh Transforms are called the Walsh Transform. Correctly, the term "Walsh Transform" refers to another notion in mathematics literature. The *Walsh Transform* has been introduced in [2] in 1950 by Fine and initially named the "Walsh-Fourier Transform" [3].

In technical applications the *Discrete Walsh Transform in Paley (or Walsh, or Hadamard) enumeration* is called the *Paley Transform* (the *Walsh Transform*, or the *Hadamard Transform*) and is denoted by PAL (or WAL, or HAD correspondingly).

By $W = (w_{kj})$ ($U = (u_{kj})$, or $H = (h_{kj})$, correspondingly) we denote the matrix of the DWT in Paley enumeration (Walsh enumeration, or Hadamard enumeration, respectively).

In mathematical books, as for instance [4], matrices W_n are introduced in the form

$$w_{kj} = w_{jk} = w_k(j/2^n), \quad 0 \leq j, k < 2^n, \tag{16.1}$$

where $\{w_k(x)\}_{k=0}^{\infty}$ – is the *Walsh-Paley system.*

The matrices U_n are introduced in the form (16.1), where $\{w_k(x)\}_{k=0}^{\infty}$ – is the *original Walsh system* [5] (Walsh system in Walsh enumeration).

The Hadamard matrices H_n were introduced in another way [6]. Let

$$H = H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then a matrix H_2 is a *Kronecker product* of H with H, thus, it is defined by

$$H_2 = H(H) = \begin{pmatrix} H & H \\ H & -H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Then, a matrix H_n is a Kronecker power

$$H_n = H^{(n)} := H(H_{n-1}) = H_{n-1}(H).$$

Matrices W_n, U_n, H_n consists of the same rows and differ only in enumeration rows.

For example,

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Let

$$\tilde{i} = (i_1\ i_2\ \ldots\ i_n)^T\ , \qquad i_k \in \{0, 1\} \tag{16.2}$$

be a *simply dyadic code* for a non-negative integer i which is less 2^n, when

$$i = \sum_{k=1}^{n} i_k \cdot 2^{k-1}. \tag{16.3}$$

Consider (see [8]) an *inverse* τ of a simply dyadic code (16.2)

$$\tau(\tilde{i}) = (i_n\ i_{n-1}\ \ldots\ i_2\ i_1)^T, \qquad \tau(i) = \sum_{j=0}^{n-1} i_{n-j} \cdot 2^j;$$

then elements of Hadamard matrix are defined as $h_{kj} = w_{\tau(k)}(j/2^n)$.

Using the *Gray code* [1], we can do analogous transition from W_n to U_n.

2. Construction of Walsh Matrices

We construct matrices W_n, U_n, H_n in another way, without using Walsh functions. We introduce *three forms of scalar product* for i, j $(0 \leq i < 2^n)$ in the form (16.3):

$$(i,j) = (i,j)_n := \sum_{k=1}^{n} i_k j_k, \; < i,j > = < i,j >_n := \sum_{k=1}^{n} i_k j_{n-k+1}, \quad (16.4)$$

$$u(i,j) = u_n(i,j) := i_1 j_n + \sum_{k=1}^{n-1} i_{k+1}(j_{n-k+1} + j_{n-k}). \quad (16.5)$$

One may consider any operations (16.4) over the field $\mathbf{Z}_2$. Then the rule (16.4) is the definition of a *quadratic form* A for construction of the Discrete Walsh Transform matrices.

In the case of DWT in Hadamard enumeration it is the unit matrix: $A = A_H := E$.

For DWT in Paley enumeration it is a matrix with unity at the secondary diagonal only (all other elements are zero).

In the case of DWT in Walsh enumeration it is a matrix with unity at the secondary diagonal and at the sub-secondary diagonal,

$$A_W := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, A_U := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Define elements of the Discrete Walsh Transform matrices by

$$w_{ij} = (-1)^{<i,j>}, \quad h_{ij} = (-1)^{(i,j)}, \quad u_{ij} = (-1)^{u(i,j)}. \quad (16.6)$$

LEMMA 16.1 *The rule (16.4) is a correct definition of the scalar product; it is a symmetric bilinear form.*

The proof is trivial.

COROLLARY 16.1 *Matrices W_n, U_n, H_n constructed by the rule (16.6) are symmetric.*

Usually [3], [4], Walsh functions are introduced as products of the Rademacher functions $r_n(x)$. By definition, $r_0(x) = (-1)^j$ for $x \in \left[\frac{j}{2}, \frac{j+1}{2}\right) = \Delta_1^j$, $j \in \{0,1\}$, and $r_0(x+1) = r_0(x)$ for $x \in [0, \infty)$.

Let $r_n(x) = r_{n-1}(2x)$ for $x \in [0, \infty)$.

REMARK 16.1 *If we consider a modified segment* $[0,1]^*$*, then we have* $\Delta_1^0 = [0^+, \frac{1}{2}^-]$*,* $\Delta_1^1 = [\frac{1}{2}^+, 1^-]$ *and so on.*

Then, the Walsh-Paley functions (for i *in the form (16.3)) is (see [3], [4])*

$$w_i(x) := \prod_{k=1}^{n} r_{k-1}^{i_k}(x).$$

By another equivalent definition, put

$$w_0(x) \equiv 1,\ w_{2^n}(x) = r_n(x)\ , w_{2^n+k}(x) = w_{2^n}(x) \cdot w_k(x) \quad \textit{for } k < 2^n.$$

A Walsh-Walsh functions (i.e., a Walsh function in ordering introduced initially by J.L. Walsh) is (see [7], [8]) $v_0 = w_0, v_1 = w_1$*,*

$$v_{2^n} = r_{n-1} \cdot r_n \quad \textit{for } n \in \mathbf{N}\ , \quad v_{2^n+k} = v_{2^n} \cdot v_k \quad \textit{for } k < 2^n. \tag{16.7}$$

The number of changes of the sign on the interval $[0,1)$ *for each function of the Walsh-Walsh system coincides with the index of the function (see [9]).*

LEMMA 16.2 *For a fixed* n*, for any* i, j *elements* w_{ij} *in the formula (16.1) and in the formula (16.6) are equal.*

Proof. For $x \in \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$, $j = \sum_{k=1}^{n} j_k \cdot 2^{k-1}$, we get $x = \sum_{k=1}^{n} \frac{j_k}{2^{n-k+1}} + x_1$ with $x_1 \in [0, 2^{-n})$. Then, $r_{n-k}(x) = (-1)^{j_k}$, or $r_{k-1}(x) = (-1)^{j_{n-k+1}}$. By using (16.1), we get $w_i(x) = \prod_{k=1}^{n} (-1)^{j_{n-k+1} i_k} = (-1)^{<i,j>}$.

LEMMA 16.3 *The number of sign changes in the* i*-th row is equal to the index* i *for the case of the DWT matrix in Walsh enumeration* $U = (u_{ij})$ *in the form (16.6).*

Proof. Denote by $l = i - i_n 2^{n-1} = \sum_{k=1}^{n-1} i_k 2^{k-1}$ and $m = [j/2] = \sum_{k=2}^{n} j_k 2^{k-1}$ the part of sums (16.3) of simply the dyadic codes for numbers i and j. We have

$$u_n(i,j) = i_1 j_n + i_2(j_n + j_{n-1}) + \ldots + i_{n-1}(j_3 + j_2) + i_n(j_2 + j_1) =$$

$$= u_{n-1}(l, m) + i_n(j_2 + j_1).$$

For the fixed index i of the row, let j runs from 0 to $2^n - 1$. Then m runs from 0 to $2^{n-1} - 1$.

The proof is by induction on n. For $n = 1$, there is nothing to prove.

We introduce a notation v_{ij} for elements of the matrix U_{n-1}, for which the inductive assumption is true. We obtain

$$u_{ij} = (-1)^{u_n(i,j)} = (-1)^{u_{n-1}(l,m)+i_n(j_2+j_1)} = v_{lm} \cdot (-1)^{i_n(j_2+j_1)}. \quad (16.8)$$

If $i_n = 0$, then we consider a upper half matrix U_n. For $i_n = 0$: this formula (16.8) make clear the independence of u_{ij} of j_1. First, by (16.8) we have $u_{ij} = v_{lm}$ and second $l = i$. It follows that the numbers of the sign changes in the i-th row of matrices U_{n-1} and U_n coincide.

If $i_n = 1$, then we consider a lower half matrix U_n. A row with an index $i = 2^{n-1} + l$ in matrix U_n corresponds to a l-th row of matrix U_{n-1} for (16.8). If $l = 0$, then the 2^{n-1}-th row has the form $1\ -1\ -1\ 1$ in period and there are 2^{n-1} changes of the sign. This is clear because by the factor $(-1)^{j_2+j_1}$ in (16.8).

The formula (16.8) corresponds to definition (16.7): the $(2^{n-1}+l)$-th row is an element-wise product of the 2^{n-1}-th row and the l-th row. Places of change of the sign for 2^{n-1}-th row and l-th row are different. Therefore, the number of changes of the sign in the row with the index $i = 2^{n-1} + l$ is equal to the index i.

COROLLARY 16.2 . *The matrix $U = (u_{ij})$ in (16.6) is the DWT matrix in Walsh enumeration in (16.1).*

In [7], Schipp have denoted $\mathbf{Z}_2$-linear rearrangements of the Walsh system. Each linear rearrangement is given by *the system of generating functions*.

For example, the Rademacher system $\{r_n(x)\}_{n=0}^{\infty}$ is a system of generating functions for the Walsh-Paley system. The system $\{R_n(x)\}_{n=0}^{\infty}$ (when $R_0 = r_0$, $R_n = r_{n-1} \cdot r_n$ as in (16.7)) is a system of generating functions for the Walsh-Walsh system. By definition in [3], *$\mathbf{Z}_2$-linear rearrangements of the Walsh system* $\{v_n(x)\}_{n=0}^{\infty}$ are

$$v_0(x) = w_0(x),\ v_{2^n}(x) = R_n(x), \quad v_{2^n+k}(x) = v_{2^n}(x) \cdot v_k(x) \quad \text{for } k < 2^n,$$

when R_n is any Walsh function such that $R_n \notin \{v_0, v_1, \ldots, v_{2^n-1}\}$.

This will be a definition of the rearrangements of Walsh system iff the map $\{w_n(x)\}_{n=0}^{\infty}$ to $\{v_n(x)\}_{n=0}^{\infty}$ is a bijection.

The rearrangement $\{v_n(x)\}$ of the Walsh system is called *regular* if $v_0(x) \equiv 1$, $v_1(x) = w_1(x)$, and the sets $\{v_k(x)\}_{k=2^n}^{2^{n+1}-1}$ and $\{w_k(x)\}_{k=2^n}^{2^{n+1}-1}$ coincide for any natural n.

Walsh-Kaczmarz system (see [3], [4]) is a regular and non-linear rearrangement of the Walsh system. In [7], Schipp called a such system as *piecewise-linear rearrangement.*

The rearrangement of 2^n initial Walsh functions in Hadamard enumeration is the linear rearrangement with the Rademacher functions r_{n-1}, r_{n-2},..., r_2, r_1, r_0 as the system of generating functions. This rearrangement is not regular.

Before we introduced the matrices A such that each row correspond to a generating function (for the rearrangement of 2^n initial Walsh functions): the indexes of the Rademacher functions are taken in the inverse order; each unity in a row has the index of the Rademacher function included into the product; this product is an ordinary generating function. Similar matrices (denoted by B) were introduced in [7], [10]; but numbers of the Rademacher functions were taken in the usual order. If we write columns of the matrix A in inverse order, we obtain the matrix B.

THEOREM 16.1 *Any non-singular matrix A of a quadratic form of order n over the field $\mathbf{Z}_2$ allows to construct a Discrete Walsh Transform matrix of order 2^n in a new ordering by means of an algorithm consisting of two steps (see (16.2), (16.3)):*

$$\begin{aligned} v(i,j) &= (\tilde{i}, A\tilde{j}) = \tilde{i}^T A\tilde{j} \quad \textit{(bilinear form)}, \\ v_{ij} &= (-1)^{v(i,j)} \quad \textit{(elements of the new matrix DWT)}. \end{aligned}$$

Conversely, any matrix DWT of order 2^n with the 0-th row consisting of 1 has a similar non-singular matrix A (matrix of a quadratic form).

A DWT matrix of order 2^n with the 0-th row consisting of 1 is symmetric iff a matrix A of a quadratic form is symmetric.

This theorem can be proven by direct calculations.

In [11], p. 26, the author introduced the definition of *infinite non-singular matrix with finite columns over a finite field.*

THEOREM 16.2 *Any infinite non-singular matrix of a quadratic form A with finite rows over the field $\mathbf{Z}_2$ allows to introduce a linear rearrangement of the Walsh system. Each row of the matrix A is a code of the ordinary generating function.*

Conversely, any linear rearrangement of the Walsh system has the similar infinite non-singular matrix A of a quadratic form.

The proof is omitted.

Denote by $M_{n,m}$ the class of $(n \times m)$ matrices.

LEMMA 16.4 *The matrix H_n in form (16.6) and Hadamard matrix in a form of the Kronecker power $H_n = H^{(n)}$ coincide for any n.*

Proof. Let $H_n = (h_{ij}^{(n)})_{i,j=0}^{2^n-1} \in M_{2^n,2^n}$. By using (16.6), we obtain

$$h_{ij} = (-1)^{(i,j)_n} = (-1)^{(m2^k, r2^k)_n} \cdot (-1)^{(l,t)_n} = (-1)^{(m,r)_{n-k}} \cdot (-1)^{(l,t)_k},$$

for $i = m2^k + l$, $j = r2^k + t$, $0 \le l, t < 2^k$, $0 \le m, r < 2^{n-k}$. This formula is

$$h_{ij}^{(n)} = h_{mr}^{(n-k)} \cdot h_{lt}^{(k)}.$$

For fixed m, r if l and t runs from 0 to $2^k - 1$, we get a block $h_{mr}^{(n-k)} \cdot H_k$ of matrix H_n represented by block form. A place of this block in Hadamards matrix corresponds to the definition of the matrix $H_{n-k}(H_k)$.

Finally, we obtain $H_n = H_{n-k}(H_k)$ for any $1 \le k < n$.

We shall introduced *a new form of the matrix product*. This product has block structure and a dimension of a block is analogous to the form and dimension of blocks in the Kronecker product.

DEFINITION 16.1 *For any matrices* $A \in M_{n,m}$, $B \in M_{k,l}$ *we denote by* $C = A\{B\} \in M_{nk,ml}$ *a block matrix of the form*

$$C = \begin{pmatrix} A^1B_1 & A^2B_1 & \dots & A^mB_1 \\ A^1B_2 & A^2B_2 & \dots & A^mB_2 \\ \vdots & \vdots & \ddots & \vdots \\ A^1B_k & A^2B_k & \dots & A^mB_k \end{pmatrix} \tag{16.9}$$

with blocks $A^j \cdot B_i \in M_{n,l}$ *such that* A^j *is a* j*-th column of the matrix* A*, and* B_i *is* i*-th row of the matrix* B.

Therefore, a new form of the matrix product is constructed by two rules for block matrices:

1 Rule 1 for enumeration of blocks - *a row of the second factor on a column of the first factor*

2 Rule 2 for the form of blocks - *a column of the first factor times a row of the second factor*.

Notice that any block A^iB_j of the matrix (16.8) is the Kronecker product

$$A^i \cdot B_j = A^i(B_j) = B_j(A^i). \tag{16.10}$$

Rows and columns of the matrix (16.9) are represented by Kronecker products of rows and columns of matrices A and B:

$$C_I = A_i(B_r) \quad \text{for} \quad I = (r-1)n + i, \tag{16.11}$$

$$C^J = B^j(A^p) \quad \text{for} \quad J = (p-1)l + j. \tag{16.12}$$

If we have enumeration of elements of a matrix from zero (but not from 1), then $I = rn + i$, $J = pl + j$.

LEMMA 16.5 *The definition of a power*

$$A^{\{d\}} = A\{A^{\{d-1\}}\} = A^{\{d-1\}}\{A\}$$

for new form of the matrix product is correct.

Proof. Let enumeration of elements of matrix begin from zero. The $(i + jn + kn^2)$-th row of a matrix $A^{\{3\}}$ have the form $A_i(A_j(A_k))$ for any i, j, k by (16.12). The $(i + jn + kn^2 + mn^3)$-th row of a matrix $A^{\{4\}}$ have the form $A_i(A_j(A_k(A_m)))$ and so on. Columns can be presented analogous.

Obviously, the Kronecker product $A(B)$ is symmetric if A and B are symmetric matrices. But, a new form of the matrix product $A\{B\}$ is not symmetric for example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA 16.6 *If matrices A and B are symmetric, then* $(A\{B\})^T = B\{A\}$.

Proof. Let to transpose a block matrix (16.9)

$$(A\{B\})^T = \begin{pmatrix} (A^1B_1)^T & (A^1B_2)^T & \dots & (A^1B_1)^T \\ \vdots & \vdots & \ddots & \vdots \\ (A^nB_1)^T & (A^nB_2)^T & \dots & (A^nB_1)^T \end{pmatrix}.$$

We get a statement by using a formula $(A^iB_j)^T = B_j^T \cdot (A^i)^T = B^jA_i$.

COROLLARY 16.3 *If the matrix A is symmetric, then the new power of matrix* $A^{\{d\}}$ *is symmetric also.*

THEOREM 16.3 *The matrix of DWT in Paley enumeration (16.1) can be defined as a new power (16.9) of the matrix* $W_1 = H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:

$$W_n = W_k\{W_{n-k}\}.$$

Proof. Let $0 \le i, j < 2^n$ are presented in the form $i = m2^k + l$, $j = r2^{n-k} + t$, for $0 \le l, r < 2^k$, $0 \le m, t < 2^{n-k}$. Then,

$$< i, j >_n = \sum_{s=1}^{k} i_s j_{n-s+1} + \sum_{s=k+1}^{n} i_s j_{n-s+1} = < l, r2^{n-k} >_n + < m2^k, t >_n,$$

or they may by presented in the form

$$< i, j >_n = < l, r >_k + < m, t >_{n-k} .$$

By (16.6) we get

$$w_{ij}^{(n)} = (-1)^{<l,r>_k} \cdot (-1)^{<m,t>_{n-k}} = w_{lr}^{(k)} \cdot w_{mt}^{(n-k)}. \qquad (16.13)$$

The first block of the matrix W_n (for $m = r = 0$) consists of elements $w_{lt}^{(n)} = w_{l0}^{(k)} \cdot w_{0t}^{(n-k)} \equiv 1$, and its block can represented as the 0-th column of the matrix W_k multiplied by the 0-th row of the matrix W_{n-k}. Any block of dimension $2^k \times 2^{n-k}$ (for fixed m, r) of the matrix W_n has the form (16.13); its block can be represented as the r-th column of the matrix W_k multiplied by m-th row of the matrix W_{n-k}, and its block is disposed as mr-th block. This is the definition of $W_k\{W_{n-k}\}$.

We recall [12] that $\|A\|_E = \sqrt{\sum_{k,l=1}^{n,m} a_{kl}^2}$ is the *Euclidean norm*. We introduced the *Euclidean orthogonality of matrices* $A, B \in M_{n,m}$: we write $A \perp B$, if $\sum_{k,l=1}^{n,m} a_{kl} \cdot b_{kl} = 0$. This definition coincides with the notion of *orthogonal vectors*, whose coordinates are all elements of the matrix in any fixed order.

We say that the collection of matrices $\{A(k)\}_{k=1}^{n \cdot m}$ constitute the *orthonormal* E*-basis* (basis for the Euclidean norm) in the class $M_{n,m}$, if $A(k) \perp A(l)$ for $k \neq l$ and $\|A(k)\|_E = 1$ for any k. For example, the standard orthonormal E-basis is a complete collection of different matrices such that all elements are equal to zero except one equal to 1.

A matrix is called *orthogonal* if $A \cdot A^T = E$ [13].

Collection of vectors $\{A_i\}$ is called *orthonormal* if $(A_i, A_j) = \delta_{ij}$ (Kronecker symbol).

Let ψ_k be rows of the DWT matrix in Paley enumeration W_n. In [10], for the case $n = 2N$, Bochkarev have solved the extremal problem:

$$\min_{\varepsilon_k = \pm 1} \| \sum_{k=0}^{2^n - 1} \varepsilon_k \psi_k \|_\infty = 2^N,$$

vectors $e_{N,j}$ are the solution of this problem.

Write the following vectors

$$e_{1,0} = (-1\ 1\ 1\ 1),\ e_{1,1} = (1\ -1\ 1\ 1),\ e_{1,2} = (1\ 1\ -1\ 1),\ e_{1,3} = (1\ 1\ 1\ -1).$$

in the form of a matrix. Then, vectors $e_{m,4i+j} = e_{1,j}(e_{m-1,i})$ are defined as Kronecker products of matrices.

Note that the solution of this extremal problem for odd n is different:

$$\min_{\varepsilon} \|W_{2N-1} \varepsilon\|_\infty = 2^N,$$

for $N = 1, 2, 3$.

By [10], the system $\{e_{N,j}\}_{j=0}^{2^n-1}$ is the total orthogonal eigenvector system for some linear rearrangement of the matrix DWT; this matrix is the Kronecker power $(W_2)^{(n-1)}$. But, this rearrangement is not the DWT matrix in Paley, Walsh or Hadamard enumeration.

By this way we get a general method of constructing the basis.

LEMMA 16.7 *A collection of vectors (in the form of the Kronecker product) $\{A_i(B_j)\}_{i,j=1}^{n,k}$ is orthonormal if both vector collections $\{A_i\}_{i=1}^{n}$, $\{B_j\}_{j=1}^{k}$ are orthonormal.*

Proof. We get for the scalar product $(A_i(B_j), A_k(B_l)) =$

$$= \sum_{r=1}^{m}(a_{ir}B_j, a_{kr}B_l) = \sum_{r=1}^{m} a_{ir}a_{kr} \cdot (B_j, B_l) = (A_i, A_k) \cdot (B_j, B_l) = \delta_{ik} \cdot \delta_{jl}.$$

By Lemma 16.7, we obtain

Proposition 1. *If matrices $A \in M_{n,n}$, $B \in M_{k,k}$ are orthogonal, then matrices $A(B)$ and $A\{B\}$ are also orthogonal. In this cases, the collection of blocks $\{A^iB_j\}_{i,j=1}^{n,k}$ in the matrix $A\{B\}$ is an orthonormal E-basis of the class $M_{n,k}$.*

Proof. Let enumeration begin from zero. A $j = (rk + s)$-th row of matrix $A(B)$ have the form $A_r(B_s)$. A $j = (rk + s)$-th column of matrix $A\{B\}$ have the form $B^s(A^r)$. By (16.10) for blocks we have proven the statement.

3. Fast Walsh Transform

In this section, previous considerations will be used to formulate the fast calculation algorithms for the discrete Walsh transform for different enumerations.

PROPOSITION 16.1 *For any $A, B \in M_{n,n}$ we have*

$$H_1(A \cdot B) = H_1(A) \cdot E_1(B) = E_1(A) \cdot H_1(B).$$

Proof. By direct calculations

$$H(A) \cdot E(B) = \begin{pmatrix} A & A \\ A & -A \end{pmatrix} \cdot \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & AB \\ AB & -AB \end{pmatrix} = H(A \cdot B).$$

Second equality can be proved in an analogous way.

Good used [13] the Proposition 16.1 for a construction of *Fast algorithms for discrete Walsh transform in Hadamard enumeration.*

COROLLARY 16.4 *This algorithm is algorithm of the* **Fast Walsh Transform (FWT)**

$$H_n = H(E_{n-1})\cdot E(H(E_{n-2}))\cdot E_2(H(E_{n-3}))\cdot \ldots \cdot E_{n-2}(H(E))\cdot E_{n-1}(H),$$

$$H_n = E_{n-1}(H)\cdot E_{n-2}(H(E))\cdot E_{n-3}(H(E_2))\cdot \ldots \cdot E(H(E_{n-2}))\cdot H(E_{n-1}).$$

REMARK 16.2 *We denote a unit matrix of order 2^n by E_n, as it is the similar notation for W_n, U_n and H_n.*

Clearly for matrices any order

$$E(A \cdot B) = E(A) \cdot E(B). \tag{16.14}$$

THEOREM 16.4 *For any $A, B \in M_{n,n}$ we have*

$$H_1\{A \cdot B\} = H_1\{A\} \cdot E_1(B).$$

Proof. We repeat the proof of the Proposition 16.1, but we permute rows of matrices $H(A)$ and $H(AB)$ in the order 0, 2^{n-1}, 1, $2^{n-1}+1$, 2, $2^{n-1}+2$, $3, \ldots$. In this way matrices $H(A)$ and $H(AB)$ convert into matrices $H\{A\}$ and $H\{AB\}$.

The submatrix of the matrix $H_1\{A\}$ consisting of even rows coincides with the upper half of the matrix $H_1(A)$. The submatrix of the matrix $H_1\{A\}$ consisting of odd rows coincides with the bottom half of the matrix $H_1(A)$.

By using the Proposition 16.1 and multiplication of matrices for a submatrix of matrix $H\{A\}$ and the matrix $E(B)$, we obtain the analogous submatrix of the matrix $H\{AB\}$.

COROLLARY 16.5 *For matrices DWT in Paley enumeration we have the recurrence*

$$W_n = H_1\{E_{n-1}\} \cdot E_1(W_{n-1}).$$

By using this relation, we get the following algorithm of **the Fast discrete Walsh transform** *in Paley enumeration*

$$W_3 = H_1\{E_2\} \cdot E_1(H_1\{E_1\}) \cdot E_2(H_1),$$

$$W_4 = H\{E_3\} \cdot E(H\{E_2\}) \cdot E_2(H\{E\}) \cdot E_3(H),$$

$$W_5 = H\{E_4\} \cdot E(H\{E_3\}) \cdot E_2(H\{E_2\}) \cdot E_3(H\{E\}) \cdot E_4(H)$$

and so on.

Proof. By using Theorems 16.3 and 16.4, we get

$$W_n = H\{W_{n-1} \cdot E_{n-1}\} = H\{E_{n-1}\} \cdot E_1(W_{n-1}).$$

In particular, $W_2 = H\{E\} \cdot E(H)$.

Combining this result and (16.14), we get $W_3 = H_1\{E_2\} \cdot E_1(W_2) =$

$$= H_1\{E_2\} \cdot E_1(H\{E\} \cdot E(H)) = H_1\{E_2\} \cdot E_1(H_1\{E_1\}) \cdot E_2(H_1).$$

Analogously we obtain

$$W_4 = H\{E_3\} \cdot E(W_3) = H\{E_3\} \cdot E(H\{E_2\} \cdot E(H\{E\}) \cdot E_2(H)),$$

$$W_4 = H\{E_3\} \cdot E(H\{E_2\}) \cdot E_2(H\{E\}) \cdot E_3(H)$$

.

EXAMPLE 16.1 *An example of this algorithm of the Fast discrete Walsh transform in Paley enumeration is the following:*

$$W_2 = H\{E\} \cdot E(W) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

$$H_1\{E_2\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$E_1(H_1\{E_1\}) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

$$E_2(H_1) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

This algorithm is more simpler than algorithms of the Fast Walsh Transform in [1], [13] and is similar to the Fast Fourier Transform.

For this reason, this result has been included in textbook [14].

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Chapter 17

AN INVERSION FORMULA FOR THE MULTIPLICATIVE INTEGRAL TRANSFORM

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Abstract An inversion formula for multiplicative integral transform with a kernel defined by characters of a locally compact zero-dimensional abelian group is obtained.

1. Introduction

In the classical trigonometrical case it is known (see [12]) that if the integral

$$\int_{-\infty}^{\infty} a(x)e^{ixy}dx,$$

where $a(x)$ is locally summable, converges everywhere to a function $f(y)$ which is finite and locally summable, then the function $a(x)$ can be recovered by the following inversion formula:

$$a(x) = (C,1)\text{ - }\lim_{h\to\infty} \frac{1}{2\pi}\int_{-h}^{h} f(y)e^{-ixy}dy \ \text{ for almost all } \ x$$

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(here by $(C, 1)$-limit of a function $\phi(h)$ defined on $(0, \infty)$, as $h \to \infty$, we mean $\lim_{h\to\infty} h^{-1} \int_0^h \phi(t)dt$).

A similar problem for the case in which the real line is replaced by a locally compact zero-dimensional abelian group and the kernel of the corresponding integral transform is defined by characters of this group was considered in [6]. Transforms of this kind are usually called multiplicative transforms (see [1]).

In this paper we consider transforms convergent to locally summable functions and in this case we obtain a generalization of the result of [6] by weakening the assumption on the type of convergence of those transforms.

The problem of getting an inversion formula for integral transform is a continual analogue of that of recovering the coefficients of a convergent series with respect to characters of compact zero-dimensional abelian group considered for example in [7].

In comparison with [4] and [6] we consider here transforms directly on the group instead of using a mapping of this group on the real line. That mapping was connected with introduction of a certain ordering in this group.

An advantage of the present new approach is that it permits to obtain some more general results on the coefficient problem and on the inversion formula which do not depend on a particular numeration (the so called Vilenkin-Palley numeration) or, respectively, on ordering of characters, as it was the case in the previous papers.

Our technique uses the notion of derivation basis and Henstock method of computing the integrals.

2. Preliminaries

Let G be a zero-dimensional locally compact abelian group which satisfies the second countability axiom. We suppose also that the group G is periodic. It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$.... \supset G_{-n} \supset ... \supset G_{-2} \supset G_{-1} \supset G_0 \supset G_1 \supset G_2 ... \supset G_n \supset ... \qquad (17.1)$$

with $G = \bigcup_{n=-\infty}^{+\infty} G_n$ and $\{0\} = \bigcap_{n=-\infty}^{+\infty} G_n$. The subgroups G_n are clopen sets with respect to this topology. As G is periodic, the factor group G_n/G_{n+1} is finite for each n and this implies that G_n (and so also all its cosets) is compact. Note that the factor group G_n/G_0 is also finite for any $n < 0$ and so the factor group G/G_0 is countable. We denote by K_n any coset of the subgroup G_n and by $K_n(g)$ the coset of the subgroup G_n which contains the element g, i.e.,

$$K_n(g) = g + G_n. \qquad (17.2)$$

For each $g \in G$ the sequence $\{K_n(g)\}$ is decreasing and $\{g\} = \bigcap_n K_n(g)$.

Now for each coset K_n of G_n we choose and fix for the rest of the paper, an element g_{K_n}. Then for each $n \in \mathbf{Z}$ we can represent any element $g \in G$ in the form:

$$g = g_{K_n} + \{g\}_n \tag{17.3}$$

where $\{g\}_n \in G_n$. We agree to put $g_{G_n} = 0$, so that $g = \{g\}_n$ if $g \in G_n$.

Let Γ denote the dual group of G, i.e., the group of characters of the group G. It is known (see [1]) that under assumption imposed on G the group Γ is also a periodic locally compact zero-dimensional abelian group (with respect to the pointwise multiplication of characters) and we can represent it as a sum of increasing sequence of subgroups

$$.... \supset \Gamma_{-n} \supset ... \supset \Gamma_{-2} \supset \Gamma_{-1} \supset \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset ... \supset \Gamma_n \supset ... \tag{17.4}$$

introducing a topology in Γ. Then $\Gamma = \bigcup_{i=-\infty}^{+\infty} \Gamma_i$ and $\bigcap_{i=-\infty}^{+\infty} \Gamma_i = \{\gamma^{(0)}\}$ where $(g, \gamma^{(0)}) = 1$ for all $g \in G$ (here and below (g, γ) denote the value of a character γ at a point g). For each $n \in \mathbf{Z}$ the group Γ_{-n} is the annulator of G_n, i.e.,

$$\Gamma_{-n} = G_n^{\perp} := \{\gamma \in \Gamma : (g, \gamma) = 1 \;\text{ for all }\; g \in G_n\}.$$

The representation (17.3), properties of a character and the definition of the annulator imply

$$(g, \gamma) = (g_{K_n}, \gamma)(\{g\}_n, \gamma) = (g_{K_n}, \gamma).$$

So with a fixed element g_{K_n}, the value (g, γ) is constant for all $g \in K_n$ and we get the following

LEMMA 17.1 *If $\gamma \in \Gamma_{-n}$ then γ is constant on each coset K_n of G_n.*

The factor groups $\Gamma_{-n-1}/\Gamma_{-n} = G_{n+1}^{\perp}/G_n^{\perp}$ and G_n/G_{n+1} are isomorphic (see [1]) and so they are of finite order for each $n \in \mathbf{Z}$. This implies that the group Γ_{-n}/Γ_0 is also finite for any $n > 0$ and Γ/Γ_0 is countable.

Now, as we have done above for the group G, we choose and fix an element $\gamma_J \in J$ for each coset J of Γ_0. Then we can represent any element $\gamma \in \Gamma$ in the form:

$$\gamma = \gamma_J \cdot \{\gamma\} \tag{17.5}$$

where $\{\gamma\} \in \Gamma_0$. We agree to put $\gamma_{\Gamma_0} = \gamma^{(0)}$, so that $\gamma = \{\gamma\}$ if $\gamma \in \Gamma_0$.

We denote by μ_G and μ_Γ the Haar measures on the groups G and Γ, respectively, and we normalize them so that $\mu_G(G_0) = \mu_\Gamma(\Gamma_0) = 1$. We can make these measures to be complete by including all the subsets of the sets of measure zero into the respective class of measurable sets.

The following property of the functions γ can be easily checked (see [8]).

LEMMA 17.2 *If* $\gamma \in \Gamma \setminus \Gamma_{-n}$ *then* $\int_{K_n}(g,\gamma)d\mu_G = 0$ *for each coset* K_n.

It follows from this lemma that if γ_1 and γ_2 are not equal identically on K_n, then they are orthogonal on K_n, i.e.,

$$\int_{K_n}(g,\gamma_1\overline{\gamma_2})d\mu_G = 0.$$

3. Integration on the Group

Now we introduce a construction of an integral on the group which covers the Lebesgue integration with respect to Haar measure on the group. This construction is based on Henstock approach to integration and on the notion of the derivation basis. To define such a derivation basis on the group G, which we denote $\mathcal{B}_G$, we take any function $\nu : G \to \mathbf{Z}$ and define a basis set by

$$\beta_\nu = \{(I,g) : g \in G, I = K_n(g), n \geq \nu(g)\}.$$

Then our basis $\mathcal{B}_G$ is the family $\{\beta_\nu\}_\nu$ where ν runs over the set of all integer-valued functions on G. In the terminology of derivation basis theory any coset K_n, $n \in \mathbf{Z}$, can be called $\mathcal{B}_G$*-interval*. We denote by $\mathcal{I}_G$ the set of all $\mathcal{B}_G$-intervals.

This basis has all the usual properties of a general derivation basis (see [9], [3]). First of all it has the *filter base property*:

- $\emptyset \notin \mathcal{B}_G$,
- for every $\beta_{\nu_1}, \beta_{\nu_2} \in \mathcal{B}_G$ there exists $\beta_\nu \in \mathcal{B}_G$ such that $\beta_\nu \subset \beta_{\nu_1} \cap \beta_{\nu_2}$ (it is enough to take $\nu = \max\{\nu_1, \nu_2\}$).

DEFINITION 17.1 *A* β_ν*-partition is a finite collection* π *of elements of* β_ν*, where the distinct elements* (I', x') *and* (I'', x'') *in* π *have* I' *and* I'' *disjoint.*

If L *is a* $\mathcal{B}_G$*-interval and* $\bigcup_{(I,x)\in\pi} I = L$ *then* π *is called* β_ν*-partition of* L.

Our basis $\mathcal{B}_G$ has the *partitioning property.* It means that the following conditions hold:

- for each finite collection $I_0, I_1, \ldots, I_n$ of $\mathcal{B}_G$-intervals with $I_1, \ldots, I_n \subset I_0$ and $I_i,\ i = 1, 2, \ldots,$ being disjoint, the difference $I_0 \setminus \bigcup_{i=1}^n I_i$ can be expressed as a finite union of pairwise disjoint $\mathcal{B}_G$-intervals;
- for each $\mathcal{B}_G$-interval L and for any $\beta_\nu \in \mathcal{B}_G$ there exists a β_ν-partition of L.

This property of $\mathcal{B}_G$ follows easily from compactness of any $\mathcal{B}_G$-interval and from the fact that any two $\mathcal{B}_G$-intervals I' and I'' are either disjoint or one of them is contained in the other one.

Note that in the case of our basis $\mathcal{B}_G$, given a point $x \in X$, any β_ν-partition contains only one pair (I, x) with this point x.

The following Henstock-type integral was defined in [8]:

DEFINITION 17.2 *Let $L \in \mathcal{I}_G$. A complex-valued function f on L is said to be Kurzweil-Henstock integrable with respect to basis $\mathcal{B}_G$ (or H_G-integrable) on L, with H_G-integral A, if for every $\varepsilon > 0$, there exists a function $\nu : L \mapsto \mathbf{Z}$ such that for any β_ν-partition π of L we have:*

$$\left| \sum_{(I,g)\in\pi} f(g)\mu_G(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_G)\int_L f$.

It is easy to check, that a function which is equal to zero almost everywhere on $L \in \mathcal{I}_G$, is H_G-integrable on L with value zero. This justifies the following extension of Definition 17.2 to the case of functions defined only almost everywhere on L.

DEFINITION 17.3 *A complex valued function f defined almost everywhere on $L \in \mathcal{I}_G$ is said to be H_G-integrable on L, with integral value A, if the function*

$$f_1(g) := \begin{cases} f(g), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise} \end{cases}$$

is H_G-integrable on L to A in the sense of Definition 17.2.

It is clear that a complex-valued function is H_G-integrable if and only if its real and imaginary parts are H_G-integrable.

REMARK 17.1 *We note that all the above definitions depend on the structure of the sequence of subgroups (17.1). So if we consider for the group Γ the definitions of the $\mathcal{B}_\Gamma$-basis and the H_Γ-integral, then we should use the sequence (17.4) in our construction.*

The *upper* and the *lower $\mathcal{B}_G$-derivative* of a set function $F : \mathcal{I}_G \mapsto I\!R$ at a point g are defined as

$$\overline{D}_G F(g) := \limsup_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}\,, \quad \underline{D}_G F(g) := \liminf_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{17.6}$$

The *$\mathcal{B}_G$-derivative* at g is

$$D_G F(g) := \lim_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{17.7}$$

It is clear that a real-valued function F is $\mathcal{B}_G$-differentiable if and only if $\overline{D}_G F(x) = \underline{D}_G F(x)$. In this case this common value is equal to $D_G F(x)$. For a complex-valued set function $F = \mathrm{Re}F + i\mathrm{Im}F$ the D_G-derivative at a point x can be defined either directly by (17.7) or as $D_G F(x) = D_G \mathrm{Re}F(x) + iD_G \mathrm{Im}F(x)$.

We say that a set function F is *$\mathcal{B}_G$-continuous* at a point g, with respect to the basis $\mathcal{B}_G$, if $\lim_{n\to\infty} F(K_n(g)) = 0$.

We note that if f is H_G-integrable on $L \in \mathcal{I}_G$ then it is H_G-integrable also on any $\mathcal{B}_G$-subinterval J of L. It can be proved that the indefinite H_G-integral on $L \in \mathcal{I}_G$ is an additive $\mathcal{B}_G$-continuous function on the set of all $\mathcal{B}_G$-subintervals of L.

The property of differentiation of the indefinite H_G-integral almost everywhere was proved in [8]:

THEOREM 17.1 *If a function f is H_G-integrable on $L \in \mathcal{I}_G$ then the indefinite H_G-integral $F(K) = (H_G)\int_K f$ as an additive function on the set of all $\mathcal{B}_G$-subintervals K of L, is $\mathcal{B}_G$-differentiable almost everywhere on L and*

$$D_G F(g) = f(g) \quad \text{a.e. on } L. \tag{17.8}$$

The following theorem related to the problem of recovering the primitive was proved in [8].

THEOREM 17.2 *Let an additive function $F : \mathcal{I}_G \to I\!R$ be $\mathcal{B}_G$-differentiable everywhere on $L \in \mathcal{I}_G$ outside of a set E with $\mu_G(E) = 0$, and $-\infty < \underline{D}_G F(x) < \overline{D}_G F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}_G$-continuous. Then the function*

$$f(x) := \begin{cases} D_G F(x), & \text{if it exists,} \\ 0, & \text{if } x \in E \end{cases}$$

is H_G-integrable on L and F is its indefinite H_G-integral.

To compare H_G-integral with the Lebesgue integral with respect to the Haar measure μ_G we use the known fact (see [1]) that the group G can be mapped on $[0, +\infty]$ by a measure preserving mapping ϕ which is one-one up to a countable set. As the Lebesgue integral is invariant under measure preserving mapping, then a function f defined on a compact subset K of G is Lebesgue integrable on K if and only if the function $f(\phi^{-1})$ is Lebesgue integrable on $\phi(K)$ with the same value of the integral. As we have mentioned in the introduction such mapping is related to introducing of certain ordering in the group G. So we have a class of mappings each of them being measure preserving but the value of the Lebesgue integral of $f(\phi^{-1})$ is the same for all ϕ from this class.

We can use now the known relation between Henstock and Lebesgue integrals on the interval of the real line. The basis $\mathcal{B}_G$ and the H_G-integral with respect to it, after mapping ϕ, are transformed into the so called P-adic basis and the P-adic Henstock integral (H_P-integral, see [6]) on the real line. As P-adic Henstock integral is known to be a generalization of the usual Henstock integral, then it is also a generalization of the Lebesgue integral. So we have

$$(L)\int_K f d\mu_G = (L)\int_{\phi(K)} f(\phi^{-1})d\mu = (H_P)\int_{\phi(K)} f(\phi^{-1}) = (H_G)\int_K f.$$

Therefore going back to the group setting we obtain that H_G-integral on G is a generalization of the Lebesgue integral on G. Hence we get

THEOREM 17.3 *If a function f is summable on $L \in \mathcal{I}_G$ with respect to μ_G then it is H_G-integrable on L and the values of the integrals coincide.*

Moreover as a consequence of Theorem 17.1 and Theorem 17.3 we obtain:

THEOREM 17.4 *If a function f is summable on $L \in \mathcal{I}_G$ with respect to μ_G then the indefinite integral $F(K) = \int_K f d\mu_G$ as an additive function on the set of all $\mathcal{B}_G$-subintervals K of L, is $\mathcal{B}_G$-differentiable almost everywhere on L and*

$$D_G F(g) = f(g) \quad \text{a.e. on} \quad L. \tag{17.9}$$

Another consequence of the Theorem 17.3 combined with Theorem 17.2 is

THEOREM 17.5 *Let an additive function $F : \mathcal{I}_G \to I\!R$ be $\mathcal{B}_G$-differentiable everywhere on $L \in \mathcal{I}_G$ outside of a set E with $\mu_G(E) = 0$, and $-\infty < \underline{D}_G F(x) < \overline{D}_G F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}_G$-continuous. Assume also that derivative $f = D_G F$ is summable on L. Then F is its indefinite Lebesgue integral.*

4. Application to the Series with Respect to the Characters

We consider here the case when the group G is compact and so the chain (17.1) is reduced to the one-side sequence

$$G = G_0 \supset G_1 \supset G_2 ... \supset G_n \supset .... \tag{17.10}$$

In this case the H_G-integral is defined on the whole group G. Moreover the group Γ of characters of the group G is discrete now (see [1]) and it can be represented as a sum of increasing chain of finite subgroups

$$\Gamma_0 \subset \Gamma_{-1} \subset \Gamma_{-2} \subset ... \subset \Gamma_{-n} \subset ... \tag{17.11}$$

where $\Gamma_0 = \{\gamma^{(0)}\}$ with $(g, \gamma^{(0)}) = 1$ for all $g \in G$.

So the characters γ constitute a countable orthogonal system on G with respect to normalized measure μ_G and we can consider a series

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma \tag{17.12}$$

with respect to this system. We define a convergence of this series at a point g as the convergence of its partial sums

$$S_n(g) := \sum_{\gamma \in \Gamma_{-n}} a_\gamma (g, \gamma) \tag{17.13}$$

when n tends to infinity.

We associate with the series (17.12) a function F defined on each coset K_n by

$$F(K_n) := \int_{K_n} S_n(g) d\mu_G. \tag{17.14}$$

This type of function is often referred to as quasi-measure (see for example [11]).

It follows easily from Lemma 17.2 that F is an additive function on the family of all $\mathcal{B}_G$-intervals.

By Lemma 17.1 the sum S_n, defined by (17.13), is constant on each K_n. Then (17.14) implies

$$S_n(g) = \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{17.15}$$

THEOREM 17.6 *The series* (17.12) *is the Fourier series of some integrable function f if and only if the function F associated with this series by expression* (17.14) *coincides on each $\mathcal{B}_G$-interval I with the indefinite integral $\int_I f$.*

Proof. This can be easily proved by the arguments used in [2, Theorem 2.8.1] for the Vilenkin-Price system.

The next two lemmas are immediate consequences of the equality (17.15).

LEMMA 17.3 *If the series* (17.12) *converges at some point $g \in G$ to a value $f(g)$ then the associated function F (see* (17.14)*) is $\mathcal{B}_G$-differentiable at g and $D_G F(g) = f(g)$. Moreover if the series* (17.12) *satisfies at a point g the conditions*

$$-\infty < \liminf_{n\to\infty} \mathrm{Re} S_n(g) \le \limsup_{n\to\infty} \mathrm{Re} S_n(g) < +\infty, \tag{17.16}$$

$$-\infty < \liminf_{n\to\infty} \mathrm{Im} S_n(g) \le \limsup_{n\to\infty} \mathrm{Im} S_n(g) < +\infty, \tag{17.17}$$

then the associated function F satisfies the inequalities

$$-\infty < \underline{D}_G \mathrm{Re} F(g) \le \overline{D}_G \mathrm{Re} F(g) < +\infty, \tag{17.18}$$

$$-\infty < \underline{D}_G \mathrm{Im} F(g) \leq \overline{D}_G \mathrm{Im} F(g) < +\infty. \tag{17.19}$$

LEMMA 17.4 *If the partial sums* (17.13) *satisfy at a point g the condition*

$$S_n(g) = o\left(\frac{1}{\mu_G(K_n(g))}\right) \tag{17.20}$$

then the associated function F is $\mathcal{B}_G$-continuous at the point g.

The next lemma gives a sufficient condition for the assumption (17.20) of the previous lemma to hold.

LEMMA 17.5 *Suppose that the coefficients* $\{a_\gamma\}$ *of a series* (17.12) *satisfy the condition*

$$\max_{\gamma \in \Gamma_{-(n+1)} \setminus \Gamma_{-n}} |a_\gamma| \to 0 \text{ if } n \to \infty, \tag{17.21}$$

then (17.20) *holds for partial sums $S_n(g)$ at each point $g \in G$.*

THEOREM 17.7 *Suppose that the partial sums* (17.13) *of the series* (17.12) *converge almost everywhere on G to a summable function f and satisfy the conditions* (17.16) *and* (17.17) *everywhere on G except on a countable set M, where* (17.20) *holds. Then* (17.12) *is the Fourier-Lebesgue series of f.*

Proof. Applying (17.15) we get that for any point g at which the series (17.12) converges to $f(g)$, the function F defined by (17.14) is $\mathcal{B}_G$-differentiable at g with $D_G F(g) = f(g)$.

According to Lemma 17.3, (17.16) and (17.17) imply inequalities (17.18) and (17.19) everywhere on G, except on the set M where by Lemma 17.4 F, together with $\mathrm{Re}F$ and $\mathrm{Im}F$, is $\mathcal{B}_G$-continuous.

Therefore, by Theorem 17.5, $\mathrm{Re}F$ and $\mathrm{Im}F$ are Lebesgue integrals of $\mathrm{Re}f$ and $\mathrm{Im}f$. Hence F is the indefinite integral of f, and using Theorem 17.6 we complete the proof.

REMARK 17.2 *In view of Lemma 17.5 we can replace the condition (17.20) by the condition (17.21) in the assumption of the above theorem.*

Let $f : G \to \mathbf{C}$ be summable on G. Then the partial sums $S_n(f, g)$ of the Fourier series of f with respect to the system of characters can be represented, according to Theorem 17.6 and formula (17.15), as

$$S_n(f, g) = \frac{1}{\mu_G(K_n(g))} \int_{K_n(g)} f.$$

From this equality together with differentiability property of the indefinite Lebesgue integral (see Theorem 17.4) follows

THEOREM 17.8 *The partial sums $S_n(f, g)$ of the Fourier series of a summable function f on G are convergent to f almost everywhere on G.*

5. The Inversion Formula for Transform in the Locally Compact Case

To simplify our notation we shall put in this section $K = K_0$, $[g] := g_K$, $\{g\} := \{g\}_0$, so that representation (17.3) with $n = 0$ for any element g of some coset K of G_0 can be rewritten in the form $g = [g] + \{g\}$ where $[g]$ is a fixed element of K and $\{g\} \in G_0$. Similarly we shall use sometimes the notation $[\gamma] := \gamma_J$ to underline duality, so the representation (17.5) for any element γ of some coset J of Γ_0 can be rewritten in the form $\gamma = [\gamma] \cdot \{\gamma\}$ where $[\gamma]$ is a fixed element of J and $\{\gamma\} \in \Gamma_0$.

Using this notation and the properties of a character γ we can write

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot (\{g\}, \{\gamma\}) \cdot ([g], \{\gamma\}). \tag{17.22}$$

Now we observe that:

1) $\{g\} \in G_0$ and $\{\gamma\} \in \Gamma_0 = G_0^{\perp}$. So $(\{g\}, \{\gamma\}) = 1$ and we can eliminate $(\{g\}, \{\gamma\})$ from representation (17.22) getting

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot ([g], \{\gamma\}). \tag{17.23}$$

2) $[\gamma] \in \Gamma_{-m(\gamma)} = G_{m(\gamma)}^{\perp}$ where $m(\gamma) \geq 0$ and $[\gamma]\lceil_{G_0}$ is a character of the subgroup G_0.

3) $([g], [\gamma])$ is constant if g belongs to a fixed coset of G_0 and γ belongs to a fixed coset of Γ_0.

4) Using the duality between G and Γ we can state that g represents a character of Γ and, similarly to the property 2), $[g]\lceil_{\Gamma_0}$ is a character of Γ_0. So $([g], \{\gamma\})$ is a value of this character at the point $\{\gamma\}$.

Therefore, according to (17.23), if g belongs to a fixed coset of G_0 and γ belongs to a fixed coset of Γ_0, we can represent (g, γ), up to a constant multiplier $([g], [\gamma])$, as a product of $(\{g\}, [\gamma])$ considered as a value of the character $[\gamma]$ at $\{g\}$, and $([g], \{\gamma\})$ considered as a value of the character $[g]$ at $\{\gamma\}$.

Now we obtain a generalization of Theorem 17.7 for a locally compact case.

THEOREM 17.9 *Assume that G is a group described in Section 2,* Γ *being its dual group. Let* $a(\gamma)$ *be a locally summable function and*

$$\lim_{n\to\infty} \int_{\Gamma_{-n}} a(\gamma)(g, \gamma)\, d\mu_\Gamma = f(g) \tag{17.24}$$

a.e. on G, where f is a locally summable function on G. Moreover everywhere on G expect a countable set T we have

$$\limsup_{n\to\infty} \left| \int_{\Gamma_{-n}} a(\gamma)(g, \gamma)\, d\mu_\Gamma \right| < +\infty. \tag{17.25}$$

and for $g \in T$ we have

$$\lim_{n\to\infty} \mu(K_n(g)) \int_{\Gamma_n} a(\gamma)(g,\gamma)d\mu = 0 \tag{17.26}$$

Then the function $a(\gamma)$ can be recovered from f by the following inversion formula:

$$a(\gamma) = \lim_{n\to\infty} \int_{G_{-n}} f(g)\overline{(g,\gamma)}d\mu_G \quad a.\ e.\ on\ \ X. \tag{17.27}$$

Proof. The proof is similar to the one of [6, theorem 9] although we use here a weaker assumption on the convergence of the integral in (17.24) and have in mind the convergence of a series as it is understood in Section 4 (see (17.13)). Having fixed a coset K suppose that $g \in K$ and let J denote any coset of Γ_0. Then by (17.23)

$$\int\limits_{\Gamma_{-n}} a(\gamma)(g,\gamma)d\mu_\Gamma = \sum_{J\subset\Gamma_{-n}} \int_J a(\gamma)(\{g\},[\gamma])\cdot([g],[\gamma])\cdot([g],\{\gamma\})d\mu_\Gamma$$

$$= \sum_{J\subset\Gamma_{-n}} (\{g\},\gamma_J)\cdot \int_J a(\gamma)([g],[\gamma])\cdot([g],\{\gamma\})d\mu_\Gamma. \tag{17.28}$$

The last sum can be considered as a partial sum

$$\sum_{J\subset\Gamma_{-n}} b_J^{(K)}(\{g\},\gamma_J) \tag{17.29}$$

of the series with respect to the system of characters $\{\gamma_J\}_J$, at the point $\{g\}$, with the coefficients

$$b_J^{(K)} = \int_J a(\gamma)([g],[\gamma])(g_K,\{\gamma\})d\mu_\Gamma.$$

According to the assumption (17.24) and the equality (17.28) this series is convergent almost everywhere on K to a function $f(g)$ which by hypothesis is summable on K.

Introducing the variable $t = \{g\} \in G_0$ we can consider this series to be convergent almost everywhere on G_0 to the summable function $p(t) = f(g_K + t)$. The partial sums (17.29) are bounded according to (17.25) and (17.28) except a countable set $M = \{t \in G_0 : g_K + t \in T\}$ where (17.26) holds which corresponds to the condition (17.20) applied to $t \in M$.

Therefore by Theorem 17.7 the coefficients $b_J^{(K)}$ are the Lebesgue-Fourier coefficients of $p(t)$, with respect to characters γ_J, i.e.,

$$b_J^{(K)} = \int_J a(\gamma)([g],[\gamma])(g_K,\{\gamma\})d\mu_\Gamma \tag{17.30}$$

$$= \int_{G_0} p(t)\overline{(\{g\}, \gamma_J)} d\mu_G = \int_K f(g)\overline{(\{g\}, \gamma_J)} d\mu_G$$

(in the last equality we use the obvious invariance of Lebesgue integral under translation). By observation 3), $([g], [\gamma])$ is constant when $g \in K$ and $\gamma \in J$ with $|([g], [\gamma])| = 1$. Hence (17.30) implies

$$\int_J a(\gamma)(g_K, \{\gamma\}) d\mu_\Gamma = \int_K f(g)\overline{([g], [\gamma])(\{g\}, \gamma_J)} d\mu_G. \tag{17.31}$$

Now we notice that for each fixed J, the value

$$\int_J a(\gamma)(g_K, \{\gamma\}) d\mu_\Gamma$$

is the Fourier coefficient, with respect to the character $\overline{g_K}$, of the summable function $a(\gamma) = a([\gamma] + \{\gamma\})$ considered as a function of $\{\gamma\} \in \Gamma_0$. Applying Theorem 17.8 to this Fourier series, we get

$$\lim_{n\to\infty} \sum_{K \subset G_{-n}} \int_J a(\gamma)(g_K, \{\gamma\}) d\mu_\Gamma \cdot \overline{(g_K, \{\gamma\})} = a([\gamma] + \{\gamma\}) = a(\gamma)$$

for almost all values of $\{\gamma\}$ on Γ_0, i.e., a.e. on J. Hence using (17.31) and (17.23) we compute

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{K \subset G_{-n}} \int_J a(\gamma)(g_K, \{\gamma\}) d\mu_\Gamma \cdot \overline{(g_K, \{\gamma\})} \\
&= \lim_{n\to\infty} \sum_{K \subset G_{-n}} \int_K f(g)\overline{([g], [\gamma])(\{g\}, \gamma_J)} d\mu_G \cdot \overline{(g_K, \{\gamma\})} \\
&= \lim_{n\to\infty} \int_{G_{-n}} f(g)\overline{(\{g\}, \gamma_J) \cdot (g_K, \{\gamma\}) \cdot ([g], [\gamma])} d\mu_G \\
&= \lim_{n\to\infty} \int_{G_{-n}} f(g)\overline{(g, \gamma)} d\mu_G = a(\gamma) \quad \text{a.e. on } J.
\end{aligned}$$

The last equality is true for any J, so we get (17.27), completing the proof.

We remark that the Theorems 17.7 and 17.9 can be generalized by the same methods to the case of any integral which is compatible with the H_G-integral (for example the analogue of the Denjoy-Perron integral on the group), but it is not true even for the Vilenkin-Price system if we use here the Denjoy-Khintchine integral (see [5]).

This kind of theorem becomes true for Denjoy-Khintchine integral if we put some additional hypothesis on the group and on the type of convergence (see [10]).

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