

Chapter 9

SUMMABILITY OF WALSH-FOURIER SERIES AND THE DYADIC DERIVATIVE

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Abstract In this survey paper we present some results on summability of one- and multi-dimensional Walsh-Fourier series and on the dyadic derivative. Three summability methods, the Fejér, Cesàro and Riesz methods are investigated. In the multi-dimensional case three types of convergence are considered, the restricted, the unrestricted and the Marcinkiewicz-type. We will prove that the maximal operator of the summability means is bounded from the martingale Hardy space H_p to L_p ($p > p_0$). For $p = 1$ we obtain a weak type inequality by interpolation, which ensures the a.e. convergence of the summability means. Similar results are formulated for the one- and multi-dimensional dyadic or Gibbs derivative. The dyadic version of the classical theorem of Lebesgue is proved, more exactly, the dyadic derivative of the dyadic integral of a function f is a.e. f .

1. Introduction

In this paper we will consider summation methods for one-dimensional and multi-dimensional Walsh-Fourier series and the one- and multi-dimensional dyadic derivative. First we present the corresponding results for trigonometric Fourier series and then the extensions to Walsh-Fourier series. Three types of summability methods will be investigated, the Fejér, Cesàro or (C, α) and the Riesz methods. The Fejér summation is a special case of the Cesàro method, $(C, 1)$ is exactly the Fejér method. In the multi-dimensional case three types of convergence and maximal operators are considered, the restricted (convergence over the diagonal or over a cone), the unrestricted (convergence over

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$\mathbb{N}^d$) and the Marcinkiewicz-type. We introduce martingale Hardy spaces H_p and prove that the maximal operators of the summability means are bounded from H_p to L_p whenever $p > p_0$ for some $p_0 < 1$. For $p = 1$ we obtain a weak type inequality by interpolation, which implies the a.e. convergence of the summability means. The a.e. convergence and the weak type inequality are proved usually with the help of a Calderon-Zygmund type decomposition lemma. However, this lemma does not work in higher dimensions. Our method, that can be applied in higher dimension, too, can be regarded as a new method to prove the a.e. convergence and weak type inequalities.

Similar results are formulated for the one- and multi-dimensional dyadic or Gibbs derivative. We get that the maximal operators are bounded from H_p to L_p if $p > p_0$ ($p_0 < 1$) and a weak type inequality if $p = 1$. This implies the dyadic version of the classical theorem of Lebesgue, more exactly, the dyadic derivative of the dyadic integral of a function f is a.e. f . In this survey paper we summarize the results appeared in this topic in the last 10–20 years. The analogous results of this paper are proved for the trigonometric, Vilenkin and Ciesielski systems in Weisz [93, 91].

2. One-dimensional Walsh-Fourier Series

The well known Carleson's theorem [8] says, that the partial sums $s_n f$ of the trigonometric Fourier series of a one-dimensional function $f \in L_2(\mathbf{T})$ converge a.e. to f as $n \rightarrow \infty$. Later Hunt [35] extended this result to all $1 < p < \infty$. This theorem does not hold, if $p = 1$. However, if we take some summability methods, we can obtain convergence for L_1 functions, too.

In 1904 Fejér [15] investigated the arithmetic means of the partial sums, the so called Fejér means and proved that if the left and right limits $f(x - 0)$ and $f(x + 0)$ exist at a point x , then the Fejér means converge to $f(x)$. One year later Lebesgue [38] extended this theorem and obtained that every integrable function is Fejér summable at each Lebesgue point, thus a.e. The Cesàro means or (C, α) ($\alpha > 0$) means are generalizations of the Fejér means; if $\alpha = 1$ then the two types of means are the same. M. Riesz [50] proved that the (C, α) ($\alpha > 0$) means $\sigma_n^\alpha f$ of a function $f \in L_1(\mathbf{T})$ converge a.e. to f as $n \rightarrow \infty$ (see also Zygmund [98, Vol. I, p.94]). Moreover, it is known that the maximal operator of the (C, α) means $\sigma_*^\alpha := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha|$ is of weak type $(1, 1)$, i.e.,

$$\sup_{\rho > 0} \rho \lambda(\sigma_*^\alpha f > \rho) \leq C \|f\|_1 \quad (f \in L_1(\mathbf{T})).$$

This result can be found implicitly in Zygmund [98, Vol. I, pp. 154-156].

For the Fejér means Móricz [43] and Weisz [78] verified that σ_*^1 is bounded from $H_1(\mathbf{T})$ to $L_1(\mathbf{T})$. The author [82] extended this result to the Cesàro summation, i.e. to σ_*^α , $\alpha > 0$ and $1/(\alpha + 1) < p < \infty$.

Stein and Weiss [66] and Butzer and Nessel [5] proved for $\gamma = 1, 2$ that the Riesz means $\sigma_n^{\alpha, \gamma} f$ of a function $f \in L_1(\mathbf{T})$ converge a.e. to f as $n \rightarrow \infty$. The author [83] verified the same result for all $\gamma \geq 1$ and that the maximal Riesz operator $\sigma_*^{\alpha, \gamma} := \sup_{n \in \mathbf{N}} |\sigma_n^{\alpha, \gamma}|$ is of weak type $(1, 1)$. Moreover, we proved in [83] that $\sigma_*^{\alpha, \gamma}$ is bounded from $H_p(\mathbf{T})$ to $L_p(\mathbf{T})$ provided that $1/(\alpha + 1) < p < \infty$ and $0 < \alpha \leq 1$. In the special case $\alpha = \gamma = 1$ the Riesz means are exactly the Fejér means.

In the next subsections analogous results will be given for Walsh-Fourier series.

Dyadic Hardy spaces

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^j$ be its Cartesian product $\mathbf{X} \times \dots \times \mathbf{X}$ taken with itself j -times. We briefly write $L_p[0, 1]^j$ instead of the space $L_p([0, 1]^j, \lambda)$ ($j \geq 1$) where λ is the Lebesgue measure.

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbf{N}$, $0 \leq k < 2^n$. Given $n \in \mathbf{N}$ and $x \in [0, 1)$ let $I_n(x)$ be the dyadic interval of length 2^{-n} which contains x . The σ -algebra generated by the dyadic intervals $\{I_n(x) : x \in [0, 1)\}$ will be denoted by $\mathcal{F}_n$ ($n \in \mathbf{N}$).

We investigate the class of *martingales* $f = (f_n, n \in \mathbf{N})$ with respect to $(\mathcal{F}_n, n \in \mathbf{N})$. The *maximal function* of a martingale f is defined by

$$f^* := \sup_{n \in \mathbf{N}} |f_n|.$$

For $0 < p \leq \infty$ the *martingale Hardy space* $H_p[0, 1)$ consists of all one-parameter martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

Recall that the Hardy and L_p spaces are equivalent, if $p > 1$, in other words,

$$H_p[0, 1) \sim L_p[0, 1) \quad (1 < p \leq \infty).$$

Moreover, the martingale maximal function is of weak type $(1, 1)$:

$$\|f\|_{H_{1, \infty}} := \sup_{\rho > 0} \rho \lambda(f^* > \rho) \leq C \|f\|_1 \quad (f \in L_1[0, 1))$$

(see Neveu [45] or Weisz [73]) and $H_1[0, 1) \subset L_1[0, 1)$.

Now some boundedness theorems for Hardy spaces are given. To this end we introduce the definition of the atoms. The *atomic decomposition* is a useful characterization of the Hardy spaces by the help of which some boundedness results, duality theorems, maximal inequalities and interpolation results can be proved. The atoms are relatively simple and easy to handle functions. If we have an atomic decomposition, then we have to prove several theorems

for atoms, only. A first version of the atomic decomposition was introduced by Coifman and Weiss [11] in the classical case and by Herz [34] in the martingale case.

A function $a \in L_\infty$ is called a p -atom if

- (a) $\text{supp } a \subset I$, $I \subset [0, 1)$ is a dyadic interval,
- (b) $\|a\|_\infty \leq |I|^{-1/p}$,
- (c) $\int_I a(x) dx = 0$.

The basic result of atomic decomposition is the following one.

THEOREM 9.1 *A martingale f is in $H_p[0, 1)$ ($0 < p \leq 1$) if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of p -atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\begin{aligned} \sum_{k=0}^{\infty} \mu_k a^k &= f \quad \text{in the sense of martingales,} \\ \sum_{k=0}^{\infty} |\mu_k|^p &< \infty. \end{aligned} \tag{9.1}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} \tag{9.2}$$

where the infimum is taken over all decompositions of f of the form (9.1).

The proof of this theorem can be found e.g. in Latter [37], Lu [39], Coifman and Weiss [11], Coifman [10], Wilson [94, 95] and Stein [65] in the classical case and in Weisz [73] for martingale Hardy spaces.

If I is a dyadic interval then let $I^r = 2^r I$ be a dyadic interval, for which $I \subset I^r$ and $|I^r| = 2^r |I|$ ($r \in \mathbb{N}$).

The following result gives a sufficient condition for V to be bounded from $H_p[0, 1)$ to $L_p[0, 1)$. For $p_0 = 1$ it can be found in Schipp, Wade, Simon and Pál [56] and in Móricz, Schipp and Wade [44], for $p_0 < 1$ see Weisz [78].

THEOREM 9.2 *Suppose that*

$$\int_{[0,1) \setminus I^r} |Va|^{p_0} d\lambda \leq C_{p_0}$$

for all p_0 -atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \leq 1$. If the sublinear operator V is bounded from $L_{p_1}[0, 1)$ to $L_{p_1}[0, 1)$ ($1 < p_1 \leq \infty$) then

$$\|Vf\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1)) \tag{9.3}$$

for all $p_0 \leq p \leq p_1$. Moreover, if $p_0 < 1$ then the operator V is of weak type $(1, 1)$, i.e. if $f \in L_1[0, 1)$ then

$$\sup_{\rho>0} \rho \lambda(|Vf| > \rho) \leq C \|f\|_1. \quad (9.4)$$

Note that (9.4) can be obtained from (9.3) by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [2] and Bennett and Sharpley [1] or Weisz [73, 91]. The interpolation of martingale Hardy spaces was worked out in [73]. Theorem 9.2 can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type $(1, 1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

We formulate also a weak version of this theorem.

THEOREM 9.3 *Suppose that*

$$\sup_{\rho>0} \rho^p \lambda\left(\{|Va| > \rho\} \cap \{[0, 1) \setminus I^r\}\right) \leq C_p$$

for all p -atoms a and for some fixed $r \in \mathbf{N}$ and $0 < p < 1$. If the sublinear operator V is bounded from L_{p_1} to L_{p_1} ($1 < p_1 \leq \infty$), then

$$\|Vf\|_{p,\infty} \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1)).$$

Walsh functions

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \quad (x \in [0, 1), n \in \mathbf{N}).$$

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*:

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k},$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad (0 \leq n_k < 2).$$

If $f \in L_1[0, 1)$ then the number

$$\hat{f}(n) := \int_{[0,1)} f w_n d\lambda \quad (n \in \mathbf{N})$$

is said to be the n th Walsh-Fourier coefficient of f . We can extend this definition to martingales as well in the usual way (see Weisz [74]). Denote by $s_n f$ the n th partial sum of the Walsh-Fourier series of a martingale f , namely,

$$s_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k.$$

It is known that $s_{2^n} f = f_n$ ($n \in \mathbf{N}$) and

$$s_{2^n} f \rightarrow f \quad \text{in } L_p\text{-norm and a.e. as } n \rightarrow \infty,$$

if $f \in L_p[0, 1)$ ($1 \leq p < \infty$).

The Carleson's theorem was extended to Walsh-Fourier series by Billard [3] and Sjölin [64]:

$$s_n f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty, \quad (9.5)$$

whenever $f \in L_p[0, 1)$ ($1 < p < \infty$). If

$$s_* f := \sup_{n \in \mathbf{N}} |s_n f|$$

denotes the maximal partial sum operator, then

$$\|s_* f\|_p \leq C_p \|f\|_p \quad (f \in L_p[0, 1), 1 < p < \infty). \quad (9.6)$$

This implies besides the a.e. convergence (9.5) also the L_p -norm convergence of $s_n f$ to f ($1 < p < \infty$). These theorems do not hold, if $p = 1$, however they can be generalized for $p = 1$ with the help of some summability methods. Fine [16] proved that the Cesàro or (C, α) means $\sigma_n^\alpha f$ ($\alpha > 0$) of a function $f \in L_1[0, 1)$ converge a.e. to f as $n \rightarrow \infty$. The convergence at all Walsh-Lebesgue points was verified by the author [72]. It is known that the maximal operator of the Fejér means ($\alpha = 1$) is of weak type $(1, 1)$, (see Schipp [52]). Fujii [19] proved that σ_*^1 is bounded from $H_1[0, 1)$ to $L_1[0, 1)$ (see also Schipp, Simon [54]). For Vilenkin-Fourier series these results are due to Simon [58]. For the Riesz means of Walsh-Fourier series it was known only that the one-dimensional maximal Riesz operator is bounded from $L_p[0, 1)$ to $L_p[0, 1)$ ($1 < p < \infty$) (see Paley [49]).

Summability of one-dimensional Walsh-Fourier series

The Fejér, Cesàro (or (C, α)) and Riesz means of a martingale f are given by

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^n s_k f = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \hat{f}(k) w_k,$$

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^n A_{n-k}^{\alpha-1} s_k f = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha \hat{f}(k) w_k$$

and

$$\sigma_n^{\alpha,\gamma} f := \frac{1}{n^{\alpha\gamma}} \sum_{k=0}^{n-1} \left((n^\gamma - k^\gamma)^\alpha \right) \hat{f}(k) w_k,$$

respectively, where

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!}.$$

If $\alpha = 1$ then $\sigma_n^\alpha f = \sigma_n f$, and so the $(C, 1)$ means are the Fejér means. Since $A_k^\alpha \sim k^\alpha$ ($k \in \mathbb{N}$), we have $A_{n-k-1}^\alpha \sim (n-k)^\alpha$. Thus the means $\sigma_n^\alpha f$ are “similar” to the ones $\sigma_n^{\alpha,1} f$.

The maximal operator of the Cesàro and Riesz means are defined by

$$\sigma_*^\alpha f := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f|, \quad \sigma_*^{\alpha,\gamma} f := \sup_{n \in \mathbb{N}} |\sigma_n^{\alpha,\gamma} f|.$$

In what follows we use a common notation σ_n for the Cesàro and Riesz means and σ_* for the corresponding maximal operators.

The next result generalizes (9.6) for the maximal operator of the summability means (see Zygmund [98] and Paley [49]).

THEOREM 9.4 *If $0 < \alpha \leq 1 \leq \gamma$ and $1 < p \leq \infty$ then*

$$\|\sigma_* f\|_p \leq C_p \|f\|_p \quad (f \in L_p[0, 1)).$$

Moreover, for all $f \in L_p[0, 1)$ ($1 < p < \infty$),

$$\sigma_n f \rightarrow f \quad \text{a.e. and in } L_p\text{-norm as } n \rightarrow \infty.$$

The L_p -norm convergence holds also, if $p = 1$. Applying Theorems 9.2 and 9.3, we extended the previous result to $p < 1$ in [74, 89, 91, 63] (for $\alpha = p = 1$ see Fujii [19]):

THEOREM 9.5 *If $0 < \alpha \leq 1 \leq \gamma$ and $1/(\alpha+1) < p \leq \infty$ then*

$$\|\sigma_* f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1))$$

and for $f \in H_{1/(\alpha+1)}[0, 1)$,

$$\|\sigma_* f\|_{1/(\alpha+1), \infty} = \sup_{\rho > 0} \rho \lambda(\sigma_* f > \rho)^{\alpha+1} \leq C \|f\|_{H_{1/(\alpha+1)}}.$$

The critical index is $p = 1/(\alpha + 1)$, if p is smaller than or equal to this critical index, then σ_* is not bounded anymore (see Simon and Weisz [63] and Simon [59]):

THEOREM 9.6 *The operator σ_* ($0 < \alpha \leq 1 \leq \gamma$) is not bounded from $H_p[0, 1)$ to $L_p[0, 1)$ if $0 < p \leq 1/(\alpha + 1)$.*

We get the next weak type $(1, 1)$ inequality from Theorem 9.5 by interpolation (Weisz [74, 89, 91], for $\alpha = 1$ Schipp [52]).

COROLLARY 9.1 *If $0 < \alpha \leq 1 \leq \gamma$ and $f \in L_1[0, 1)$ then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_* f > \rho) \leq C \|f\|_1.$$

Since the set of the Walsh polynomials is dense in $L_1[0, 1)$, Corollary 9.1 and the usual density argument (see Marcinkiewicz, Zygmund [41]) imply

COROLLARY 9.2 *If $0 < \alpha \leq 1 \leq \gamma$ and $f \in L_1[0, 1)$ then*

$$\sigma_n f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty.$$

Recall that this convergence result was proved by Fine [16] for the (C, α) summation and in this general version by the author [74, 89]. With the help of the conjugate functions we proved also

THEOREM 9.7 *If $0 < \alpha \leq 1 \leq \gamma$ and $1/(\alpha + 1) < p \leq \infty$ then*

$$\|\sigma_n f\|_{H_p} \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1)).$$

COROLLARY 9.3 *If $0 < \alpha \leq 1 \leq \gamma$, $1/(\alpha + 1) < p < \infty$ and $f \in H_p[0, 1)$ then*

$$\sigma_n f \rightarrow f \quad \text{in } H_p\text{-norm as } n \rightarrow \infty.$$

Some of these theorems can be found in Gát [21] and Simon [60, 59] for the Walsh-Kaczmarz system, in Simon [58] and Weisz [80] for the Vilenkin systems.

Note that for $\alpha > 1$ the results can be reduced to the $\alpha = 1$ case.

3. The Dyadic Derivative

The one-dimensional differentiation theorem due to Lebesgue

$$f(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt \quad \text{a.e.} \quad (f \in L_1[0, 1))$$

is well known (see e.g. Zygmund [98]).

In this section the dyadic analogue of this result will be formulated. Gibbs [26], Butzer and Wagner [6, 7] introduced the concept of the *dyadic derivative* as follows. For each function f defined on $[0, 1)$ set

$$(\mathbf{d}_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1} (f(x) - f(x \dot{+} 2^{-j-1})), \quad (x \in [0, 1)).$$

Then f is said to be *dyadically differentiable* at $x \in [0, 1)$ if $(\mathbf{d}_n f)(x)$ converges as $n \rightarrow \infty$. It was verified by Butzer and Wagner [7] that every Walsh function is dyadically differentiable and

$$\lim_{n \rightarrow \infty} (\mathbf{d}_n w_k)(x) = k w_k(x) \quad (x \in [0, 1), k \in \mathbf{N}).$$

Let W be the function whose Walsh-Fourier coefficients satisfy

$$\hat{W}(k) := \begin{cases} 1, & \text{if } k = 0 \\ 1/k, & \text{if } k \in \mathbf{N}, k \neq 0. \end{cases}$$

The *dyadic integral* of $f \in L_1[0, 1)$ is introduced by

$$\mathbf{I}f(x) := f * W(x) := \int_0^1 f(t) W(x \dot{+} t) dt.$$

Notice that $W \in L_2[0, 1) \subset L_1[0, 1)$, so $\mathbf{I}$ is well defined on $L_1[0, 1)$.

Let the *maximal operator* be defined by

$$\mathbf{I}_* f := \sup_{n \in \mathbf{N}} |\mathbf{d}_n(\mathbf{I}f)|.$$

The boundedness of $\mathbf{I}_*$ from $L_p[0, 1)$ to $L_p[0, 1)$ ($1 < p \leq \infty$) is due to Schipp [51]:

THEOREM 9.8 *If $1 < p < \infty$ then*

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_p \quad (f \in L_p[0, 1)).$$

Schipp and Simon [54] verified that $\mathbf{I}_*$ is bounded from $L \log L[0, 1)$ to $L_1[0, 1)$. Recall that $L \log L[0, 1) \subset H_1[0, 1)$. These results are extended to $H_p[0, 1)$ spaces in the next theorem (see Weisz [81]).

THEOREM 9.9 *Suppose that $f \in H_p[0, 1) \cap L_1[0, 1)$ and*

$$\int_0^1 f(x) dx = 0. \tag{9.7}$$

Then

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_{H_p}$$

for all $1/2 < p < \infty$.

We get by interpolation

COROLLARY 9.4 *If $f \in L_1[0, 1)$ satisfies (9.7), then*

$$\sup_{\rho > 0} \rho \lambda(\mathbf{I}_* f > \rho) \leq C \|f\|_1.$$

The dyadic analogue of the Lebesgue's differentiation theorem follows easily from the preceding weak type inequality:

COROLLARY 9.5 *If $f \in L_1[0, 1)$ satisfies (9.7), then*

$$\mathbf{d}_n(\mathbf{I}f) \rightarrow f \quad \text{a.e., as } n \rightarrow \infty.$$

Corollaries 9.4 and 9.5 are due to Schipp [51] (see also Weisz [76]).

4. More-dimensional Walsh-Fourier Series

The analogue of the Carleson's theorem does not hold in higher dimensions. However, the summability results above can be generalized for the more-dimensional case. For multi-dimensional trigonometric Fourier series Zygmund [98] verified that if $f \in L(\log L)^{d-1}(\mathbf{T}^d)$ then the Cesàro means $\sigma_n^\alpha f$ converges to f a.e. and if $f \in L_p[0, 1)^d$ ($1 \leq p < \infty$) then $\sigma_n^\alpha f \rightarrow f$ in $L_p[0, 1)^d$ norm as $\min(n_1, \dots, n_d) \rightarrow \infty$. Moreover, if n must be in a cone then the a.e. convergence holds for all $f \in L_1(\mathbf{T}^d)$. More exactly, Marcinkiewicz and Zygmund [41] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbf{T}^d)$ converge a.e. to f as $\min(n_1, \dots, n_d) \rightarrow \infty$ provided that n is in a positive cone, i.e., provided that $2^{-\tau} \leq n_i/n_j \leq 2^\tau$ for every $i, j = 1, \dots, d$ and for some $\tau \geq 0$ ($n = (n_1, \dots, n_d) \in \mathbf{N}^d$).

In the next subsections Walsh- and multi-dimensional analogues of the results above will be given.

d -dimensional dyadic Hardy spaces

By a *dyadic rectangle* we mean a Cartesian product of d dyadic intervals. For $n \in \mathbf{N}^d$ and $x \in [0, 1)^d$ let $I_n(x) := I_{n_1}(x_1) \times \dots \times I_{n_d}(x_d)$, where $n = (n_1, \dots, n_d)$ and $x = (x_1, \dots, x_d)$. The σ -algebra generated by the dyadic rectangles $\{I_n(x) : x \in [0, 1)^d\}$ will be denoted again by $\mathcal{F}_n$ ($n \in \mathbf{N}^d$).

For d -parameter martingales $f = (f_n, n \in \mathbf{N}^d)$ with respect to $(\mathcal{F}_n, n \in \mathbf{N}^d)$ we introduce three kinds of maximal functions and Hardy spaces. The *maximal functions* are defined by

$$f^\circ := \sup_{n \in \mathbf{N}} |f_n|, \quad f^* := \sup_{n \in \mathbf{N}^d} |f_n|,$$

where $\mathbf{n} := (n, \dots, n) \in \mathbf{N}^d$ for $n \in \mathbf{N}$. In the first maximal function we have taken the supremum over the diagonal, in the second one over $\mathbf{N}^d$. Let E_n denote the conditional expectation operator with respect to $\mathcal{F}_n$. Obviously, if $f \in L_1[0, 1]^d$ then $(E_n f, n \in \mathbf{N}^d)$ is a martingale. In the third maximal function the supremum is taken over $d - 1$ indices: for fixed x_i we define

$$f^i(x) := \sup_{n_k \in \mathbf{N}, k=1, \dots, d; k \neq i} |E_{n_1} \dots E_{n_{i-1}} E_{n_{i+1}} \dots E_{n_d} f(x)|.$$

For $0 < p \leq \infty$ the *martingale Hardy spaces* $H_p^\circ[0, 1]^d$, $H_p[0, 1]^d$ and $H_p^i[0, 1]^d$ consists of all d -parameter martingales for which

$$\|f\|_{H_p^\circ} := \|f^\circ\|_p < \infty, \quad \|f\|_{H_p} := \|f^*\|_p < \infty, \quad \|f\|_{H_p^i} := \|f^i\|_p < \infty,$$

respectively. One can show (see Weisz [73]) that $L(\log L)^{d-1}[0, 1]^d \subset H_1^i[0, 1]^d \subset H_{1,\infty}[0, 1]^d$ ($i = 1, \dots, d$), more exactly,

$$\|f\|_{H_{1,\infty}} := \sup_{\rho > 0} \rho \lambda(f^* > \rho) \leq C \|f\|_{H_1^i} \quad (f \in H_1^i[0, 1]^d)$$

and

$$\|f\|_{H_1^i} \leq C + C \| |f| (\log^+ |f|)^{d-1} \|_1 \quad (f \in L(\log L)^{d-1}[0, 1]^d)$$

where $\log^+ u = 1_{\{u > 1\}} \log u$. Moreover, it is known that

$$H_p^\circ[0, 1]^d \sim H_p[0, 1]^d \sim H_p^i[0, 1]^d \sim L_p[0, 1]^d \quad (1 < p < \infty).$$

The hardy spaces $H_p^\circ[0, 1]^d$. To obtain some convergence results of the summability means over the diagonal we consider the Hardy space $H_p^\circ[0, 1]^d$. Now the situation is similar to the one-dimensional case.

A function $a \in L_\infty[0, 1]^d$ is a *cube p -atom* if

- (a) $\text{supp } a \subset I$, $I \subset [0, 1]^d$ is a dyadic cube,
- (b) $\|a\|_\infty \leq |I|^{-1/p}$,
- (c) $\int_I a(x) dx = 0$.

The basic result of atomic decomposition is the following one (see Weisz [73, 91]).

THEOREM 9.10 *A d -parameter martingale f is in $H_p^\circ[0, 1]^d$ ($0 < p \leq 1$) if and only if there exist a sequence $(a^k, k \in \mathbf{N})$ of cube p -atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\begin{aligned} \sum_{k=0}^{\infty} \mu_k a^k &= f \quad \text{in the sense of martingales,} \\ \sum_{k=0}^{\infty} |\mu_k|^p &< \infty. \end{aligned} \tag{9.8}$$

Moreover,

$$\|f\|_{H_p^\circ[0,1]^d} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} \tag{9.9}$$

where the infimum is taken over all decompositions of f of the form (9.8).

For a rectangle $R = I_1 \times \dots \times I_d \subset \mathbf{R}^d$ let $R^r := 2^r R := I_1^r \times \dots \times I_d^r$ ($r \in \mathbf{N}$). The following result generalizes Theorem 9.2.

THEOREM 9.11 *Suppose that*

$$\int_{[0,1]^d \setminus I^r} |Va|^{p_0} d\lambda \leq C_{p_0}$$

for all cube p_0 -atoms a and for some fixed $r \in \mathbf{N}$ and $0 < p_0 \leq 1$. If the sublinear operator V is bounded from $L_{p_1}[0,1]^d$ to $L_{p_1}[0,1]^d$ ($1 < p_1 \leq \infty$) then

$$\|Vf\|_p \leq C_p \|f\|_{H_p^\circ} \quad (f \in H_p^\circ[0,1]^d) \tag{9.10}$$

for all $p_0 \leq p \leq p_1$. Moreover, if $p_0 < 1$ then the operator V is of weak type $(1,1)$, i.e. if $f \in L_1[0,1]^d$ then

$$\sup_{\rho>0} \rho \lambda(|Vf| > \rho) \leq C \|f\|_1. \tag{9.11}$$

Again, (9.11) follows from (9.10) by interpolation.

The Hardy spaces $H_p[0,1]^d$. In the investigation of the convergence in the Prigheim's sense (i.e. over all n) we use the Hardy spaces $H_p[0,1]^d$. The atomic decomposition for $H_p[0,1]^d$ is much more complicated. One reason of this is that the support of an atom is not a rectangle but an open set. Moreover, here we have to choose the atoms from $L_2[0,1]^d$ instead of $L_\infty[0,1]^d$. This atomic decomposition was proved by Chang and Fefferman [9, 14] and Weisz [85, 91]. For an open set $F \subset [0,1]^d$ denote by $\mathcal{M}(F)$ the maximal dyadic subrectangles of F .

A function $a \in L_2$ is a p -atom if

- (a) $\text{supp } a \subset F$ for some open set $F \subset [0,1]^d$,

- (b) $\|a\|_2 \leq |F|^{1/2-1/p}$,
- (c) a can be further decomposed into the sum of “elementary particles”
 $a_R \in L_2$, $a = \sum_{R \in \mathcal{M}(F)} a_R$ in L_2 , satisfying
- (d) $\text{supp } a_R \subset R \subset F$,
- (e) for all $i = 1, \dots, d$ and $R \in \mathcal{M}(F)$ we have

$$\int_{[0,1)} a_R(x) dx_i = 0,$$

- (f) for every disjoint partition $\mathcal{P}_l$ ($l = 1, 2, \dots$) of $\mathcal{M}(F)$,

$$\left(\sum_l \left\| \sum_{R \in \mathcal{P}_l} a_R \right\|_2^2 \right)^{1/2} \leq |F|^{1/2-1/p}.$$

THEOREM 9.12 *A d -parameter martingale f is in $H_p[0, 1)^d$ ($0 < p \leq 1$) if and only if there exist a sequence $(a^k, k \in \mathbf{N})$ of p -atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\begin{aligned} \sum_{k=0}^{\infty} \mu_k a^k &= f \quad \text{in the sense of martingales,} \\ \sum_{k=0}^{\infty} |\mu_k|^p &< \infty. \end{aligned} \tag{9.12}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}$$

where the infimum is taken over all decompositions of f of the form (9.12).

The corresponding results to Theorems 9.2 and 9.11 for the $H_p[0, 1)^d$ space are much more complicated. First we consider the two-dimensional case. Since the definition of the p -atom is very complex, to obtain a usable condition about the boundedness of the operators, we have to introduce simpler atoms.

If $d = 2$, a function $a \in L_2[0, 1)^2$ is called a *simple p -atom*, if

- (a) $\text{supp } a \subset R$, $R \subset [0, 1)^2$ is a dyadic rectangle,
- (b) $\|a\|_2 \leq |R|^{1/2-1/p}$,
- (c) $\int_{[0,1)} a(x) dx_i = 0$ for $i = 1, 2$.

Note that $H_p[0, 1)^d$ cannot be decomposed into rectangle p -atoms, a counterexample can be found in Weisz [73]. However, the following result says that

for an operator V to be bounded from $H_p[0, 1)^d$ to $L_p[0, 1)^d$ ($0 < p \leq 1$) it is enough to check V on simple p -atoms and the boundedness of V on $L_2[0, 1)^d$.

THEOREM 9.13 *Suppose that $d = 2$, $0 < p_0 \leq 1$ and there exists $\eta > 0$ such that*

$$\int_{[0,1)^2 \setminus R^r} |Va|^{p_0} d\lambda \leq C_{p_0} 2^{-\eta r}, \quad (9.13)$$

for all simple p_0 -atoms a and for all $r \geq 1$. If the sublinear operator V is bounded from $L_2[0, 1)^d$ to $L_2[0, 1)^d$, then

$$\|Vf\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1)^d) \quad (9.14)$$

for all $p_0 \leq p \leq 2$. In particular, if $p_0 < 1$ then the operator V is of weak type $(H_1^i[0, 1)^d, L_1[0, 1)^d)$, i.e. if $f \in H_1^i[0, 1)^d$ for some $i = 1, \dots, d$ then

$$\sup_{\rho > 0} \rho \lambda(|Vf| > \rho) \leq C \|f\|_{H_1^i}. \quad (9.15)$$

Inequality (9.15) follows from (9.14) by interpolation. In some sense the space $H_1^i[0, 1)^d$ plays the role of the one-dimensional $L_1[0, 1)$ space.

Theorem 9.13 for two-dimensional classical Hardy spaces is due to Fefferman [14] and for martingale Hardy spaces to Weisz [79]. Journé [36] verified that the preceding result do not hold for dimensions greater than 2. So there are fundamental differences between the theory in the two-parameter and three- or more-parameter cases. Now we present the analogous theorem for higher dimensions.

Let $d \geq 3$. A function $a \in L_2[0, 1)^d$ is called a *simple p -atom*, if there exist dyadic intervals $I_i \subset [0, 1)$, $i = 1, \dots, j$ for some $1 \leq j \leq d - 1$ such that

- (a) $\text{supp } a \subset I_1 \times \dots \times I_j \times A$ for some measurable set $A \subset [0, 1)^{d-j}$,
- (b) $\|a\|_2 \leq (|I_1| \dots |I_j| |A|)^{1/2-1/p}$,
- (c) $\int_{I_i} a(x) x_i dx_i = \int_A a d\lambda = 0$ for $i = 1, \dots, j$.

Of course if $a \in L_2[0, 1)^d$ satisfies these conditions for another subset of $\{1, \dots, d\}$ than $\{1, \dots, j\}$, then it is also called simple p -atom.

As in the two-parameter case, $H_p[0, 1)^d$ cannot be decomposed into simple p -atoms. It is easy to see that condition (9.13) can also be formulated as follows:

$$\int_{(I_1^c)^c \times I_2} |Va|^{p_0} d\lambda + \int_{(I_1^c)^c \times I_2^c} |Va|^{p_0} d\lambda \leq C_{p_0} 2^{-\eta r}$$

and the corresponding inequality holds for the dilation of I_2 , where H^c denotes the complement of the set H and $R = I_1 \times I_2$. For higher dimensions we generalize this form. The next theorem is due to the author [85, 91].

THEOREM 9.14 Let $d \geq 3$. Suppose that the operators V_n are linear for every $n \in \mathbf{N}^d$ and

$$V := \sup_{n \in \mathbf{N}^d} |V_n|$$

is bounded on $L_2[0, 1]^d$. Suppose that there exist $\eta_1, \dots, \eta_d > 0$, such that for all simple p_0 -atoms a and for all $r_1, \dots, r_d \geq 1$

$$\int_{(I_1^{r_1})^c \times \dots \times (I_j^{r_j})^c} \int_A |Va|^{p_0} d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \dots 2^{-\eta_j r_j}.$$

If $j = d - 1$ and $A = I_d \subset [0, 1]$ is a dyadic interval, then we assume also that

$$\int_{(I_1^{r_1})^c \times \dots \times (I_{d-1}^{r_{d-1}})^c} \int_{(I_d)^c} |Va|^{p_0} d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \dots 2^{-\eta_{d-1} r_{d-1}}.$$

Then

$$\|V^* f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1]^d)$$

for all $p_0 \leq p \leq 2$. In particular, if $p_0 < 1$ and $f \in H_1^i[0, 1]^d$ for some $i = 1, \dots, d$ then

$$\sup_{\rho > 0} \rho \lambda(|Vf| > \rho) \leq C \|f\|_{H_1^i}. \quad (9.16)$$

More-dimensional Walsh Functions

The Kronecker product $(w_n, n \in \mathbf{N}^d)$ of d Walsh systems is said to be the d -dimensional Walsh system. Thus

$$w_n(x) := w_{n_1}(x_1) \dots w_{n_d}(x_d)$$

where $n = (n_1, \dots, n_d) \in \mathbf{N}^d$, $x = (x_1, \dots, x_d) \in [0, 1]^d$.

The n th Fourier coefficient of $f \in L_1[0, 1]^d$ is introduced by

$$\hat{f}(n) := \int_{[0, 1]^d} f w_n d\lambda \quad (n \in \mathbf{N}^d).$$

With the usual extension of Fourier coefficients to martingales we can define the n th partial sum of the Walsh-Fourier series of a martingale f by

$$s_n f := \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \hat{f}(k) w_k, \quad (n \in \mathbf{N}^d).$$

Under $\sum_{j=1}^d \sum_{k_j=0}^{n_j-1}$ we mean the sum $\sum_{k_1=0}^{n_1-1} \dots \sum_{k_d=0}^{n_d-1}$.

It is known that $s_{2^{n_1}, \dots, 2^{n_d}} f = f_n$ ($n \in \mathbf{N}^d$) and

$$s_{2^{n_1}, \dots, 2^{n_d}} f \rightarrow f \quad \text{in } L_p\text{-norm as } n \rightarrow \infty,$$

if $f \in L_p[0, 1]^d$ ($1 \leq p < \infty$). If $p > 1$ then the convergence holds also a.e. Moreover,

$$s_n f \rightarrow f \quad \text{in } L_p\text{-norm as } n \rightarrow \infty,$$

whenever $f \in L_p[0, 1]^d$ ($1 < p < \infty$) (see e.g. Schipp, Wade, Simon and Pál [56]). The a.e. convergence of $s_n f$ is not true (Fefferman [12, 13]). However, investigating the partial sums over the diagonal, only, we have the following results (Móricz [42] or Schipp, Wade, Simon and Pál [56]):

$$\| \sup_{n \in \mathbf{N}} |s_n f| \|_2 \leq C \|f\|_2 \quad (f \in L_2[0, 1]^d)$$

and for $f \in L_2[0, 1]^d$

$$s_n f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty \quad (n \in \mathbf{N}). \quad (9.17)$$

In contrary to the trigonometric case, it is unknown whether this result holds for functions in $L_p[0, 1]^d$, $1 < p < 2$.

Summability of d -dimensional Walsh-Fourier series

The *Fejér*, *Cesàro* and *Riesz means* of a martingale f are defined by

$$\sigma_n f := \frac{1}{\prod_{i=1}^d n_i} \sum_{j=1}^d \sum_{k_j=1}^{n_j} s_k f = \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \prod_{i=1}^d \left(1 - \frac{k_i}{n_i}\right) \hat{f}(k) w_k,$$

$$\begin{aligned} \sigma_n^\alpha f &:= \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^d \sum_{k_j=1}^{n_j} A_{n_j-k_j}^{\alpha_j-1} s_k f \\ &= \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \left(\prod_{i=1}^d A_{n_i-k_i-1}^{\alpha_i} \right) \hat{f}(k) w_k \end{aligned}$$

and

$$\sigma_n^{\alpha, \gamma} f := \frac{1}{\prod_{i=1}^d n_i^{\alpha_i \gamma_i}} \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \left(\prod_{i=1}^d (n_i^{\gamma_i} - k_i^{\gamma_i})^{\alpha_i} \right) \hat{f}(k) w_k,$$

respectively. We use again the common notation σ_n for the Cesàro and Riesz means. For a given $\tau \geq 0$ the *restricted* and *non-restricted maximal operators* are defined by

$$\sigma_\circ f := \sup_{\substack{2^{-\tau} \leq n_i/n_j \leq 2^\tau \\ i, j=1, \dots, d}} |\sigma_n f|, \quad \sigma_* f := \sup_{n \in \mathbf{N}^d} |\sigma_n f|.$$

The next result follows easily from Theorem 9.4 by iteration.

THEOREM 9.15 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$) and $1 < p \leq \infty$ then*

$$\|\sigma_* f\|_p \leq C_p \|f\|_p \quad (f \in L_p[0, 1]^d).$$

Moreover, for all $f \in L_p[0, 1]^d$ ($1 < p < \infty$),

$$\sigma_n f \rightarrow f \quad \text{a.e. and in } L_p\text{-norm as } n \rightarrow \infty.$$

The L_p -norm convergence holds also, if $p = 1$. Here $n \rightarrow \infty$ means that $\min(n_1, \dots, n_d) \rightarrow \infty$ (the Pringsheim's sense of convergence).

Restricted summability. In this subsection we investigate the operator $\sigma_\circ$ and the convergence of $\sigma_n f$ over the cone $\{n \in \mathbf{N}^d : 2^{-\tau} \leq n_i/n_j \leq 2^\tau; i, j = 1, \dots, d\}$, where $\tau \geq 0$ is fixed.

THEOREM 9.16 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$) and*

$$p_0 := \max\{1/(\alpha_j + 1), j = 1, \dots, d\} < p < \infty,$$

then

$$\|\sigma_\circ f\|_p \leq C_p \|f\|_{H_p^\circ} \quad (f \in H_p^\circ[0, 1]^d).$$

This theorem for $\alpha_1 = \alpha_2 = 1$ and for two-dimensional functions was proved by the author [75]. The general version of Theorem 9.16 can be found in Weisz [84, 91] (see also Goginava [27]).

For the Fejér means (i.e. $\alpha_j = \gamma_j = 1, j = 1, \dots, d$) there are counterexamples for the boundedness of $\sigma_\circ$ if $p \leq p_0 = 1/2$ (Goginava [33]).

THEOREM 9.17 *The operator $\sigma_\circ^1$ ($\alpha_j = 1, j = 1, \dots, d$) is not bounded from $H_p^\circ[0, 1]^d$ to $L_p[0, 1]^d$ if $0 < p \leq 1/2$.*

By interpolation we obtain ([84])

COROLLARY 9.6 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$) and $f \in L_1[0, 1]^d$ then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_\circ f > \rho) \leq C \|f\|_1.$$

The set of the Walsh polynomials is dense in $L_1[0, 1]^d$, so Corollary 9.6 imply the Walsh analogue of the Marcinkiewicz-Zygmund result.

COROLLARY 9.7 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$) and $f \in L_1[0, 1]^d$ then*

$$\sigma_n f \rightarrow f \quad \text{a.e.}$$

as $n \rightarrow \infty$ and $2^{-\tau} \leq n_i/n_j \leq 2^\tau$ ($i, j = 1, \dots, d$).

Note that this corollary is due to the author [75, 84], for Fejér means and for two-dimensional functions it can also be found in Gát [20].

The following results are known ([84]) for the norm convergence of $\sigma_n f$.

THEOREM 9.18 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$) and $p_0 < p < \infty$, then*

$$\|\sigma_n f\|_{H_p^\circ} \leq C_p \|f\|_{H_p^\circ} \quad (f \in H_p^\circ[0, 1)^d)$$

whenever $2^{-\tau} \leq n_i/n_j \leq 2^\tau$ ($i, j = 1, \dots, d$).

COROLLARY 9.8 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$), $p_0 < p < \infty$ and $f \in H_p^\circ$ then*

$$\sigma_n f \rightarrow f \quad \text{in } H_p^\circ\text{-norm}$$

as $n \rightarrow \infty$ and $2^{-\tau} \leq n_i/n_j \leq 2^\tau$ ($i, j = 1, \dots, d$).

Unrestricted summability. Now we deal with the operator σ_* and the convergence of $\sigma_n f$ as $n \rightarrow \infty$, i.e. $\min(n_1, \dots, n_d) \rightarrow \infty$. The next result is due to the author ([79, 87, 85, 92]).

THEOREM 9.19 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$) and*

$$p_0 := \max\{1/(\alpha_j + 1), j = 1, \dots, d\} < p < \infty,$$

then

$$\|\sigma_* f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1)^d).$$

THEOREM 9.20 (GOGINAVA [33]) *The operator σ_*^1 ($\alpha_j = 1, j = 1, \dots, d$) is not bounded from $H_p[0, 1)^d$ to $L_p[0, 1)^d$ if $0 < p \leq 1/2$.*

By interpolation we get here a.e. convergence for functions from the spaces $H_1^i[0, 1)^d$ instead of $L_1[0, 1)^d$.

COROLLARY 9.9 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ and $f \in H_1^i[0, 1)^d$ ($i, j = 1, \dots, d$) then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_* f > \rho) \leq C \|f\|_{H_1^i}.$$

Recall that $H_1^i[0, 1)^d \supset L(\log L)^{d-1}[0, 1)^d$ for all $i = 1, \dots, d$.

COROLLARY 9.10 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ and $f \in H_1^i[0, 1)^d$ ($i, j = 1, \dots, d$) then*

$$\sigma_n f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty.$$

Gát [23, 24] proved for the Fejér means that this corollary does not hold for all integrable functions.

THEOREM 9.21 *The a.e. convergence is not true for all $f \in L_1[0, 1)^d$.*

THEOREM 9.22 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$) and $p_0 < p < \infty$, then*

$$\|\sigma_n f\|_{H_p} \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1)^d, n \in \mathbf{N}^d).$$

COROLLARY 9.11 *If $0 < \alpha_j \leq 1 \leq \gamma_j$ ($j = 1, \dots, d$), $p_0 < p < \infty$ and $f \in H_p$ then*

$$\sigma_n f \rightarrow f \quad \text{in } H_p\text{-norm as } n \rightarrow \infty.$$

5. More-dimensional Dyadic Derivative

The multi-dimensional version of Lebesgue's differentiation theorem reads as follows:

$$f(x) = \lim_{h \rightarrow 0} \frac{1}{\prod_{j=1}^d h_j} \int_{x_1}^{x_1+h_1} \dots \int_{x_d}^{x_d+h_d} f(t) dt \quad \text{a.e.,}$$

if $f \in L(\log L)^{d-1}[0, 1)^d$. If $\tau^{-1} \leq |h_i/h_j| \leq \tau$ for some fixed $\tau \geq 0$ and all $i, j = 1, \dots, d$, then it holds for all $f \in L_1[0, 1)^d$. To present the dyadic version of this result we introduce first the *multi-dimensional dyadic derivative* ([4]) by the limit of

$$\begin{aligned} (\mathbf{d}_n f)(x) &:= \sum_{i=1}^d \sum_{j_i=0}^{n_i-1} 2^{j_1+\dots+j_d-d} \\ &\quad \times \sum_{\varepsilon_i=0}^1 (-1)^{\varepsilon_1+\dots+\varepsilon_d} f(x_1 + \varepsilon_1 2^{-j_1-1}, \dots, x_d + \varepsilon_d 2^{-j_d-1}). \end{aligned}$$

The *d-dimensional dyadic integral* is defined by

$$\mathbf{I}f(x) := f*(W \times \dots \times W)(x) = \int_0^1 \dots \int_0^1 f(t) W(x_1 + t_1) \dots W(x_d + t_d) dt$$

and for given $\tau \geq 0$ let the *maximal operators* be

$$\mathbf{I}_\circ f := \sup_{|n_i - n_j| \leq \tau, i, j=1, \dots, d} |\mathbf{d}_n(\mathbf{I}f)|, \quad \mathbf{I}_* f := \sup_{n \in \mathbf{N}^d} |\mathbf{d}_n(\mathbf{I}f)|.$$

THEOREM 9.23 *Suppose that $f \in H_p^\circ[0, 1)^d \cap L_1[0, 1)^d$ and*

$$\int_0^1 f(x) dx_i = 0 \quad (i = 1, \dots, d). \quad (9.18)$$

Then

$$\|\mathbf{I}_\circ f\|_p \leq C_p \|f\|_{H_p^\circ}$$

for all $d/(d+1) < p < \infty$.

COROLLARY 9.12 If $f \in L_1[0, 1]^d$ satisfies (9.18), then

$$\sup_{\rho>0} \rho \lambda(\mathbf{I}_\circ f > \rho) \leq C \|f\|_1.$$

COROLLARY 9.13 If $\tau \geq 0$ is arbitrary and $f \in L_1[0, 1]^d$ satisfies (9.18), then

$$\mathbf{d}_n(\mathbf{I}f) \rightarrow f \quad \text{a.e., as } n \rightarrow \infty \text{ and } |n_i - n_j| \leq \tau.$$

Theorem 9.23 and Corollaries 9.12 and 9.13 are due to the author [76, 91]. The two corollaries were also shown by Gát [22].

For the operator $\mathbf{I}_*$ the following results were verified in Weisz [77, 88, 91].

THEOREM 9.24 If (9.18) is satisfied and $1/2 < p < \infty$ then

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p[0, 1]^d).$$

COROLLARY 9.14 If $f \in H_1^i[0, 1]^d$ ($i = 1, \dots, d$) satisfies (9.18), then

$$\sup_{\rho>0} \rho \lambda(\mathbf{I}_* f > \rho) \leq C \|f\|_{H_1^i}.$$

COROLLARY 9.15 If $f \in H_1^i[0, 1]^d (\supset L(\log L)^{d-1}[0, 1]^d)$ ($i = 1, \dots, d$) satisfies (9.18), then

$$\mathbf{d}_n(\mathbf{I}f) \rightarrow f \quad \text{a.e., as } n \rightarrow \infty.$$

Note that this result for $f \in L \log L$ is due to Schipp and Wade [55] in the two-dimensional case.

Similarly to the dyadic derivative we can define the Vilenkin derivative (see Onneweer [46]) and one can prove similar results (see Pál and Simon [47, 48], Gát and Nagy [25] and Simon and Weisz [62, 61]).

6. Marcinkiewicz-Cesàro summability of Walsh-Fourier Series

As we have seen in (9.17), the diagonal partial sums $s_n f$ converge to f a.e. as $n \rightarrow \infty$. In this section we take the arithmetic means $\tau_n f$ (the so called Marcinkiewicz-Fejér means) of the sequence $(s_n f)$ and present some a.e. results for $\tau_n f$ and inequalities.

Marcinkiewicz [40] verified that the Marcinkiewicz-Fejér (or -Cesàro) means $\tau_n f$ of the two-dimensional trigonometric Fourier series of a function $f \in$

$L \log L(\mathbf{T}^2)$ converge a.e. to f as $n \rightarrow \infty$. Later Zhizhiashvili [96, 97] extended this result to all $f \in L_1(\mathbf{T}^2)$. Besides this convergence the author [86] proved that the maximal operator $\tau_* f := \sup_{n \in \mathbf{N}} |\tau_n f|$ is bounded from $H_p^\circ[0, 1]^d$ to $L_p[0, 1]^d$ and it is of weak type $(1, 1)$, where $p_0 < p < \infty$ and $p_0 < 1$.

Let

$$\tau_n f := \tau_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^n A_{n-k}^{\alpha-1} s_k f$$

be the Marcinkiewicz-Cesàro means and

$$\tau_* f := \sup_{n \in \mathbf{N}} |\tau_n f|.$$

The following results were proved by Weisz [90, 91] and Goginava [28, 29, 30, 32].

THEOREM 9.25 *If $0 < \alpha \leq 1$ and $d/(d + \alpha) < p < \infty$ then*

$$\|\tau_* f\|_p \leq C_p \|f\|_{H_p^\circ} \quad (f \in H_p^\circ[0, 1]^d).$$

THEOREM 9.26 (GOGINAVA [31]) *The operator τ_*^1 is not bounded from $H_p^\circ[0, 1]^d$ to $L_p[0, 1]^d$ if $0 < p \leq d/(d + 1)$.*

COROLLARY 9.16 *If $0 < \alpha \leq 1$ and $f \in L_1[0, 1]^d$ then*

$$\sup_{\rho > 0} \rho \lambda(\tau_* f > \rho) \leq C \|f\|_1.$$

COROLLARY 9.17 *If $0 < \alpha \leq 1$ and $f \in L_1[0, 1]^d$ then*

$$\tau_n f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty.$$

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