

Chapter 2

GIBBS DERIVATIVES - THE DEVELOPMENT OVER 40 YEARS IN CHINA

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Abstract Beginning at 1975, we learned about Gibbs derivatives, since then, we have studied the topics in the Gibbs derivatives, some related properties of this new kind of derivatives and their applications.

In this survey, we arrange the presentation as follows.

1. Generalize Gibbs derivatives to p -adic groups, p_j -adic groups, local fields and locally compact Vilenkin groups; apply to approximation theory by using Gibbs derivatives;
2. Define function spaces on local fields described by Gibbs derivatives;
3. Show comparison between $\mathbf{R}^n$ analysis based on classical calculus and local field analysis based on Gibbs calculus;
4. Establish principles for defining "rate of change";
5. View some applications of Gibbs derivatives to fractal analysis.

1. Generalizations of Gibbs Derivatives and Applications

Since the Gibbs derivatives were born in 1967, many mathematicians and physicists, engineers and technicians have paid great attentions to this topic. And so do we.

We learned about the Gibbs derivatives in 1975. Since then, we have started the study of Gibbs derivatives. We have generalized Gibbs derivatives to p -adic groups, p_j -adic groups, local fields and locally compact Vilenkin groups.

Gibbs derivatives on p -adic groups and p_j -adic groups

In 1978, [1], F.X. Ren, W.Y. Su, W.X. Zheng generalized the dyadic Gibbs derivatives to p -adic case for p as a prime.

Let for

$$G = \{x = (x_{-s}, x_{-s+1}, \dots, x_{-1}, x_0, x_1, \dots) : x_j \in \{0, 1, \dots, p-1\}, s \in \mathbf{P}\}$$

be a p -adic group, with addition coordinately $\oplus \bmod p$ and $\mathbf{P} = \{0\} \cup \mathbf{N}$. If we endow a topology $\tau = \{B_k\}_{k \in \mathbf{Z}}$ to G , where

$$B_k = \{x = (x_k, x_{k+1}, \dots) \in G : x_j \in \{0, 1, \dots, p-1\}, j \geq k, x_k \neq 0\}$$

is a k -neighborhood of $0 \in G$, then G becomes a non-discrete, totally disconnected, locally compact topological group; and the Walsh system

$$\Gamma = G^\wedge = \{w_p(x, y) = \exp \frac{2\pi i}{p} x \otimes y : x, y \in G\}, \quad x \otimes y = \sum x_{1-j} y_j$$

is the character group of G .

If $f : G \rightarrow \mathbf{C}$ is a complex valued function on G , we call

$$\begin{aligned} f^{<1>}(x) &= \lim_{N \rightarrow +\infty} \Delta_N f(x) \\ &= \lim_{N \rightarrow +\infty} \sum_{k=-N}^N p^k \sum_{j=0}^{p-1} A_j f(x \oplus jp^{-k-1}), \quad x \in G \end{aligned} \quad (2.1)$$

p -adic pointwise Gibbs derivative of $f(x)$ at $x \in G$, where

$$A_0(p) = \frac{p-1}{2}, A_j = \frac{\omega^j}{1-\omega^j}, \quad j = 1, \dots, p-1, \quad \omega = \exp \frac{2\pi i}{p}.$$

One also can define p -adic strong Gibbs derivative of $f(x)$ in $L^r(G)$, $1 \leq r \leq +\infty$.

The special case of (2.1) is for the compact case, $G_0 \subset G$ is the compact subgroup of G

$$G_0 = \{x = (x_0, x_1, \dots) \in G : x_j \in \{0, 1, \dots, p-1\}, j \in \mathbf{P}\},$$

and

$$\Gamma_0 = G_0^\wedge = \{w_p(k, x) = \exp \frac{2\pi i}{p} k \otimes x : x \in G, k \in \mathbf{P}\}.$$

Then the Gibbs derivatives of $f(x)$ at $x_0 \in G_0$, just take the first sum in (2.1) as $\sum_{k=0}^N$, and the Gibbs derivative has the form:

$$\begin{aligned} f^{<1>}(x) &= \lim_{N \rightarrow +\infty} \Delta_N f(x) \\ &= \lim_{N \rightarrow +\infty} \sum_{k=0}^N p^k \sum_{j=0}^{p-1} A_j f(x \oplus jp^{-k-1}), \quad x \in G_0. \end{aligned}$$

In 1983, Z.L.He [10] generalized Gibbs derivatives to the p_j -adic groups in fractional forms for $j \in \mathbf{P}$ in $L^r(G_0)$ strong sense

$$D^{<\alpha>} * f(x) = \left[\sum_{k=0}^{p_j-1} k^\alpha \overline{\psi_k} \right] * f(x), \quad x \in G_0, \quad (2.2)$$

where $\{\psi_k(x)\}_{k=0}^{+\infty}$ is p_j -adic Walsh system.

In 1988, He [24] generalized Gibbs derivatives again to the p_j -adic groups in fractional forms for $j \in \mathbf{P}$ in $L^r(G)$ strong sense for the locally compact case $G = \mathbf{R}, \mathbf{T}, \mathbf{Z}$, $1 < r < +\infty$.

From 1981 to 1990, we completed the following:

- 1 Generalizations of the dyadic Gibbs derivatives to that of p -adic [1] case and p_j -adic [11], [25] cases;
- 2 Proved important properties of Gibbs derivatives, including
 - (a) Operation properties [1]-[5],[7],[10],[11],[24];
 - (b) Fourier-Walsh transform properties (also distribution sense) [5], [7],[23],[28];
 - (c) Approximation properties (Jackson and Bernstein theorems) [5], [8]-[10],[12],[24],[28];
 - (d) Construction of some approximation identity kernels, such as, Abel-Poisson type kernels [6],[20], a class of approximation identity kernels [13]-[15], Vallee-Poussin kernels [30], and so on [39].

Gibbs derivatives on local fields

Since the dyadic analysis is a special one of p -adic case, which is a special case of local fields, so we have paid our attention to the study of analysis on local fields from 1984.

Let K be a locally compact, totally disconnected, complete topological field with non-archimedean norm, $x \rightarrow |x|$ a mapping from K to $\mathbf{R}^+$, such that:

- (i) $|x| \geq 0, |x| = 0 \iff x = 0$;
- (ii) $|xy| = |x||y|$;
- (iii) $|x + y| \leq \max\{|x|, |y|\}$.

Thus it has a topological base $\tau = \{G_k\}_{k \in \mathbf{Z}}$ of $0 \in K$ as

$$G_k = \{x \in K : |x| \leq q^{-k}\}, \quad k \in \mathbf{Z},$$

with

(a) $\{G_k\}_{k \in \mathbf{Z}}$ is strictly decreasing, $\cdots \subset G_{k+1} \subset G_k \subset G_{k-1} \subset \cdots$, $k \in \mathbf{Z}$, every G_k is closed, open and compact set;

(b) $\bigcup_{k=-\infty}^{+\infty} G_k = K$, and $\bigcap_{k=-\infty}^{+\infty} G_k = \{0\}$, and $q = p^c$, $c \in \mathbf{N}$, p is a prime.

Moreover, let Γ be the character group of K , then the annihilators of G_k is

$$\Gamma_k = \{\xi \in \Gamma : |\xi| \leq p^k\}, \quad k \in \mathbf{Z}$$

with

(a') $\{\Gamma_k\}_{k \in \mathbf{Z}}$ is strictly increasing, $\cdots \subset \Gamma_{k-1} \subset \Gamma_k \subset \Gamma_{k+1} \subset \cdots$, $k \in \mathbf{Z}$, every Γ_k is closed, open and compact set;

(b') $\bigcup_{k=-\infty}^{+\infty} \Gamma_k = \Gamma$, and $\bigcap_{k=-\infty}^{+\infty} \Gamma_k = \{0\}$.

In 1985, [18], W.X. Zheng generalized the Gibbs derivatives to local fields. He still uses the limit to define the Gibbs derivatives on K :

$$\begin{aligned} f^{<1>}(x) &= \lim_{N \rightarrow +\infty} \Delta_N f(x) \\ &= \lim_{N \rightarrow +\infty} \sum_{l=-N+t}^{N+t} q^{-n-j+1} \sum_{l=0}^{q^N-1} \sum_{v=0}^{p-1} \exp\left(\frac{-2\pi i}{p}\right) f(x + l\beta^{-j}) \end{aligned} \quad (2.3)$$

for a fixed $t \in \mathbf{N}$, $q = p^c$, $c \in \mathbf{N}$, p is a prime. Later, he gave some important properties in [15]-[17], [19], [21], [22], [25]-[27], [31]-[34].

In 1993, [38], H.K. Jiang generalized again the Gibbs derivatives to a -adic group for

$$a = \{\cdots, a_{-n}, \cdots, a_{-1}, a_0, a_1, \cdots, a_n, \cdots\}, \quad a_j \geq 1$$

by also a limit form.

Then, in 1992, [35], W.Y. Su gave the definitions of Gibbs derivatives and Gibbs integrals by the so called pseudo-differential operators:

For a Haar measurable function $f : K \rightarrow \mathbf{C}$, if the integral

$$T_{<\cdot>^m} f(x) = \int_{\Gamma} \left\{ \int_K <\xi>^m f(t) \overline{\chi_{\xi}}(t-x) dt \right\} d\xi, \quad m \geq 0 \quad (2.4)$$

exists at $x \in K$ with $<\xi> = \max\{1, |\xi|\}$, then $T_{<\cdot>^m} f(x)$ is called the pointwise Gibbs derivative of f at $x \in K$ with order m , denoted by

$$f^{<m>}(x) = T_{<\cdot>^m} f(x), \quad m \geq 0. \quad (2.5)$$

Also we can define the $L^r(K)$ -strong Gibbs derivatives with order m .

And for $m \leq 0$, it is the pointwise Gibbs integral of f at $x \in K$ with order m , denoted by

$$f_{<m>}(x) = T_{<\cdot>^m} f(x), \quad m \leq 0. \quad (2.6)$$

Many important properties of Gibbs derivative $f^{<m>}(x) = T_{<\cdot>^m} f(x)$, $m \geq 0$ and Gibbs integral $f_{<m>}(x) = T_{<\cdot>^m} f(x)$, $m \leq 0$ are shown in [35], [36], [40]-[43], including the operation properties, approximation operators and approximation properties, all using the Gibbs derivatives.

2. Function Spaces

Not only in "harmonic analysis", "differential equations", but also in the "fractal analysis", lots of function spaces play very important roles, so we have to establish some function spaces and study properties by using Gibbs derivatives.

Besov type spaces, Triebel B -type spaces and F -type spaces

In 1988, W.Y. Su studied the boundness of the pseudo-differential operators in Besov type spaces on a local field [26]. As special cases, she got Lebesgue type spaces, Sobolev spaces in [26].

In 1989, C.W. Onneweer and W.Y. Su defined the homogeneous Besov spaces on the Vilenkin groups, studied some properties about dual spaces and got many interesting and useful results [29].

In 1992, G.C. Zhou and W.Y. Su defined the Triebel B -type spaces $B_{p,q}^s(K_n)$ and F -type spaces $F_{p,q}^s(K_n)$ on n -dimension local field K_n , proved some embedding theorems, and lifting properties [37].

Holder type spaces $C^\alpha(K)$

In 2006, [46], W.Y. Su gave the definition of the Holder spaces, and proved a very interesting property: the Holder spaces exactly describe the Gibbs smoothness.

In fact, the Holder type space $C^\sigma(K)$, $\sigma \in [0, \infty)$ is defined just by the Littlewood-Paley decomposition [46], and we proved that

THEOREM 2.1 *The Holder type space $C^\sigma(K)$, $\sigma \in [0, \infty)$ has the following properties:*

- (1) if $f \in C^\sigma(K)$, then for $\forall 0 \leq \lambda \leq \sigma$, function f has the Gibbs-type derivative $T_{<\cdot>^\lambda} f(x)$, $x \in K$, and $T_{<\cdot>^\lambda} f \in C^{\sigma-\lambda}(K)$;
- (2) if $T_{<\cdot>^\sigma} f \in C^0(K)$, then for $\forall 0 \leq \lambda \leq \sigma$, function f has the Gibbs-type derivative $T_{<\cdot>^{\sigma-\lambda}} f(x)$, $x \in K$, and $T_{<\cdot>^{\sigma-\lambda}} f \in C^\lambda(K)$.

Lipschitz classes $Lip(\alpha, K)$

In 2007, "Lipschitz classes on local fields" has published, some essential properties are included in it [49].

For $0 < \alpha < +\infty$, we call the function class

$$Lip(\alpha, K) = \{f \in C(K) : \|f(\cdot + h) - f(\cdot)\|_{C(K)} = O(|h|^\alpha)\}$$

the Lipschitz class on local field K . And the relationship between the Holder type spaces and the Lip classes are also reviled:

THEOREM 2.2 *In a local field K , we have*

$$Lip(\alpha, K) = C^\alpha(K), \quad \alpha \in (0, +\infty).$$

3. Comparison Between the Classical Derivatives and Gibbs Derivatives

In the case of the Euclidean space $\mathbf{R}^n$ as an underline space, we use

$$C^m \equiv C^m(\mathbf{R}^n), \quad m \in \mathbf{P}$$

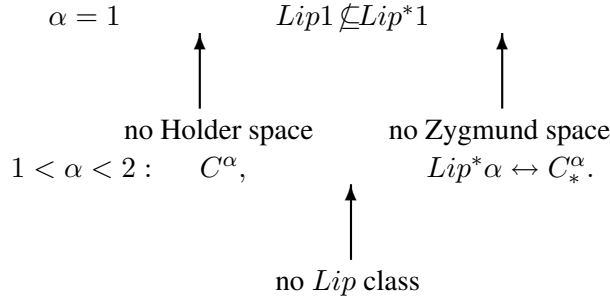
to denote the function space of m -order continuous differential functions. To describe the smoothness of functions defined on $\mathbf{R}$, we list as follows [49]

$$\begin{array}{ccccccc} \dots & \not\subseteq & C^{m+1} & \not\subseteq & C^m & \not\subseteq & C^1 & \not\subseteq & Lip1 & \not\subseteq & Lip\alpha & \not\subseteq & Lip\beta & \not\subseteq & \dots & C \\ & & \uparrow & & & & & & & & \uparrow & & & & \\ & & m \in \{2, 3, \dots\} & & & & & & & & 1 > \alpha > \beta > 0 & & & & \end{array}$$

(2.7)

We note that, there is a "gap" between $Lip\alpha$ and $Lip^*\alpha$; moreover, also has a "gap" for the Holder space C^α , $\alpha \in (0, +\infty) \setminus \mathbf{N}$, and Zygmund space C_*^α , $\alpha \in (0, +\infty) \setminus \mathbf{N}$:

$$0 < \alpha < 1 : C^\alpha \leftrightarrow Lip\alpha = Lip^*\alpha \leftrightarrow C_*^\alpha,$$



However, in the case for that of local field K as an underline space, " p -type smoothness" is described by Holder type space $C^\sigma(K)$, $\sigma \geq 0$ fully by Theorem 1 in [46], no needed 2-order difference:

$$C^\alpha(K) \leftrightarrow Lip(\alpha, K), \quad \alpha \in (0, +\infty). \quad (2.8)$$

Moreover, no "gap", there is a corresponding nice relationship

$$\begin{array}{ccc}
 C^\alpha(K) & \subset & C^\beta(K), \quad 0 \leq \beta < \alpha < +\infty \\
 \updownarrow & & \updownarrow \\
 Lip\alpha & \subset & Lip\beta
 \end{array}$$

and it shows that the space with more higher smoothness is contained in that of the lower one.

We see the essential differences between the analysis of $\mathbf{R}$ and K .

(1) the algebraic structure of $(\mathbf{R}, +, \times)$ and $(K, \oplus, \otimes)$ are totally different;

(2) the topological structures of $(\mathbf{R}, +, \times, \tau)$ and $(K, \oplus, \otimes, |\cdot|)$ are totally different: $\mathbf{R}$ is connected, and K is totally disconnected;

(3) the structures of the corresponding character groups of $\mathbf{R}$ and K are totally different: by the dual theory, $\Gamma_{\mathbf{R}} \leftrightarrow \mathbf{R}(\Gamma_{[-1,1]} \leftrightarrow \mathbf{T})$, $\Gamma_K \leftrightarrow K(\Gamma_{G_0} \leftrightarrow \{0\} \cup \mathbf{N})$, thus $\Gamma_{\mathbf{R}}$ and Γ_K are connected and totally disconnected, respectively;

(4) the character equations (eigen-equations) of $\mathbf{R}$ and K are totally different: they are $y' = \lambda y$ and $y^{<1>} = \lambda y$ respectively, where y' is the classical derivative of y on $\mathbf{R}$, which is "the rate of change at $x \in \mathbf{R}$ ", and $y^{<1>}$ is the Gibbs derivative of y on K , which describes "a global rate of change on K ", respectively; and the non-zero values λ (eigen-values), which make character equations have non-zero solutions, are $\lambda = iy$ and $\lambda = y$, respectively; and the non-zero solutions are character functions $\exp 2\pi i y x$ and $\exp \frac{2\pi i y \odot x}{p}$, respectively;

(5) the best approximation equivalent theorems on $\mathbf{R}$ and K are totally different: in fact, we have for Euclidean space as an underline space

$$\begin{aligned} 0 < \alpha < 1 : \quad & f \in Lip\alpha \Leftrightarrow E_n(f, C_{[0,2\pi]}) = O(n^{-\alpha}); \\ \alpha = 1 : \quad & f \in Lip1 \Rightarrow E_n(f, C_{[0,2\pi]}) = O(n^{-1}), \\ & f \in Lip^*1 \Leftrightarrow E_n(f, C_{[0,2\pi]}) = O(n^{-1}); \end{aligned}$$

where $E_n(f, C_{[0,2\pi]})$ is the best approximation of function $f \in C_{[0,2\pi]}$. However, for the case of local fields, on the compact subgroup $G_0 = D = \{x \in K : |x| \leq 1\}$, the following 4 terms are equivalent for $\alpha > 0, r \in \{0\} \cup \mathbf{N}$

$$\begin{aligned} (i) \quad & f^{<r>} \in Lip(\alpha, G_0); \\ (ii) \quad & \omega(f^{<r>}, p^{-n}, C(G_0)) = O(p^{-n\alpha}), \quad n \rightarrow \infty; \\ (iii) \quad & E_{p^n}(f, C(G_0)) = O(p^{-n(\alpha+\gamma)}), \quad n \rightarrow \infty; \\ (iv) \quad & \|f(\cdot) - S_{p^n}(f, \cdot)\|_{C(G_0)} = O(p^{-n(\alpha+\gamma)}), \quad n \rightarrow \infty. \end{aligned}$$

The similar results for $K^+ = K$ hold for the local field K [3]: for α and r , $\alpha > 0, r \in \{0\} \cup \mathbf{N}$, the following statements are equivalent

$$\begin{aligned} (i) \quad & f^{<r>} \in Lip(\alpha, K); \\ (ii) \quad & \omega(f^{<r>}, p^{-n}, C(K)) = O(p^{-n\alpha}), \quad n \rightarrow \infty; \\ (iii) \quad & E_{p^n}(f, C(K)) = O(p^{-n(\alpha+\gamma)}), \quad n \rightarrow \infty. \end{aligned}$$

Thus, the 1-order continuity modulus determines the equivalent theorems on a local field K , so does $Lip\alpha$ class.

We have analyzed the properties of the group operations, topological structures, character groups, rate of changes (thus, the classical derivatives and Gibbs derivatives, as well as eigen-equations and eigen-values), and the approximation structures of $\mathbf{R}$ and K . So we can see that the formulas (2.7) and (2.8) describe the smoothness of functions defined on $\mathbf{R}$ and K , respectively, and we also may understand that: "classical derivatives" is suitable for the analysis on $\mathbf{R}$ and "Gibbs smoothness" is suitable for the analysis on local fields.

Moreover, we need introduce Lip^* class on $\mathbf{R}^n$ since the equivalent theorems need 2-order continuity modulus, and just need Lip class on K , and have $C^\alpha(K) \leftrightarrow Lip(\alpha, K)$, since we just need 1-order continuity modulus.

Recall the Theorem 1.1 in [13]: If $f \in L^1$, then for each $t \in (-\infty, +\infty)$, it holds

$$\lim_{p \rightarrow \infty} \int_0^{+\infty} f(x) \bar{\omega}_p(t, x) dx = \int_0^{+\infty} f(x) \exp[-2\pi i(t x - (sngt)\{|t|\}\{x\})] dx,$$

where $\{x\}$ is the decimal part of a real number x , $\omega_p(t, x)$ is the p -series Walsh function. This theorem not only shows that the essential difference between the

analysis on the Euclidean spaces and local fields, but also proves that: kinds of results of the Fourier analysis on local fields, are never the special cases of those on Euclidean spaces when $p \rightarrow \infty$, thus the guesswork is denied: results in Fourier analysis over local fields are just special cases of those over Euclidean spaces.

Moreover, we may connect this fact with the famous result in fractal geometry: increase the numbers of sides of the Koch curve, the angle 60° can never be disappeared.

Thus, we may interpreter that: $\mathbf{R}^n$ analysis is a powerful tool to describe the universe in the macroscopic point of view, and the analysis over local fields is suitable to serve to the point of view in microcosmic spaces.

4. Principles for Defining "rate of change"

In the classical case, Newton's derivatives have a meaning "rate of change", so we may ask: does it have the similar meaning for the Gibbs derivatives? In other words, does it have some principles for defining operations which have the meaning "rate of change"? It seems to have as follows.

(1) Differentiation has an inverse operator - the integral operator

$$\frac{d}{dx} \int f(x) dx = f(x), \quad \int df(x) = f(x);$$

(2) Derivatives have Fourier transform formula - Fourier transform of $f'(x)$

$$[f'(\cdot)]^\wedge(\xi) = i\xi f^\wedge(\xi);$$

(3) Differentiation has approximation properties - direct and inverse theorems, such as

$$E_n(f, L_r(R)) = O(n^{-\alpha-s}) \iff f^{<s>} \in Lip(\alpha, L_r(R));$$

(4) Differentiation satisfies eigen-equations - eigen-functions, eigen-values

eigen-equations

$$\frac{dy}{dx} = \lambda y;$$

eigen-values

$$\lambda = i\xi; \quad \xi \in \mathbf{R};$$

eigen-functions

$$e^{i\xi x}, x \in \mathbf{R}, \quad \xi \in \mathbf{R}$$

(5) Function spaces which differential functions live — spaces on $\mathbf{R}$ shch as $C^n, C^\alpha, C_*^\alpha$.

Then for Gibbs derivatives, the above five essential properties all have been proved (see [35]). Thus we may conclude that: Gibbs derivatives make sense for describing the rate of change for lots of natural phenomena, and are certainly very suitable for scientific studies in many scientific fields.

5. Applications to Fractal Analysis by Gibbs Derivatives

Fractals, no classical derivatives, such as the Wererstrass function, it is a typical example nowhere has derivative, and it is also a typical example in

fractal analysis. Other examples of fractal: The Brownian motions, Cantor set, Julia set, Kock curve, and so on.

How to describe the rate of change of a fractal?

We get more and more idea that Gibbs derivatives are nice tools to describe the rete of changes for these functions which have no classical derivatives.

Recently, we have some papers to study the p -adic Gibbs derivatives on local fields, such as the Cantor functions, the Weierstress-like functions, Weierstress type functions, and so on, see [47],[48],[50]-[54].

In [47], H. Qiu, W.Y. Su show that for the Weierstrass-like function on dyadic local field K_2 as

$$g(x) = \begin{cases} \sum_{j=1}^{+\infty} x_j (\frac{1}{2})^j, & \forall x \in B_1, \\ 0, & \text{otherwise} \end{cases}$$

where $B_1 = \{x \in K_2 : |x| \leq 2^{-1}\}$, $\text{supp} g \subset B_1$, the prime element $\beta \in K_2$ with non- archimedean norm $|\beta| = 2^{-1}$, $x_j \in \{0, 1\}$, $j = 1, 2, \dots$.

THEOREM 2.3 *The function $g(x)$ is infinitely integrable; and is an m -order differentiable with $0 < m < 1$, and*

$$g^{<m>}(x) = \begin{cases} \frac{1}{4} + \frac{2^m}{4} - \frac{2^{2m-2}}{1-2^{m-1}} + \sum_{j=1}^{+\infty} x_j (\frac{1}{2})^{j-(j+1)m}, & |x| \leq 2^{-1}, \\ \frac{1}{4} - \frac{2^m}{4}, & |x| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, there is no 1-order derivatives at any point in B_1 .

In [52], we consider the 3-adic Cantor function $f(x)$ on 3-series local field K_3 with

$$x \in K_3 \Rightarrow x = \sum_{j=-s}^{+\infty} x_j \beta^j, \quad x_j \in \{0, 1, 2\}, j = -s, -s+1, \dots, s \in \mathbf{Z},$$

with $|\beta| = 3^{-1}$.

And we define 3-adic Cantor function as:

$$f(x) = \begin{cases} \sum_{j=0}^{k-2} (x_j - 1) \cdot (\frac{1}{2})^{j+1} + (\frac{1}{2})^k, & x \in D, x_{k-1} = 0; \\ & x_j \neq 0, 0 \leq j \leq k-2; \\ \sum_{j=0}^{+\infty} (x_j - 1) \cdot (\frac{1}{2})^{j+1}, & x \in D, x_j \neq 0, 0 \leq j < +\infty; \\ 0, & x \notin D, \end{cases}$$

where $D = \{x \in K_3 : |x| \leq 1\}$.

Then, we have

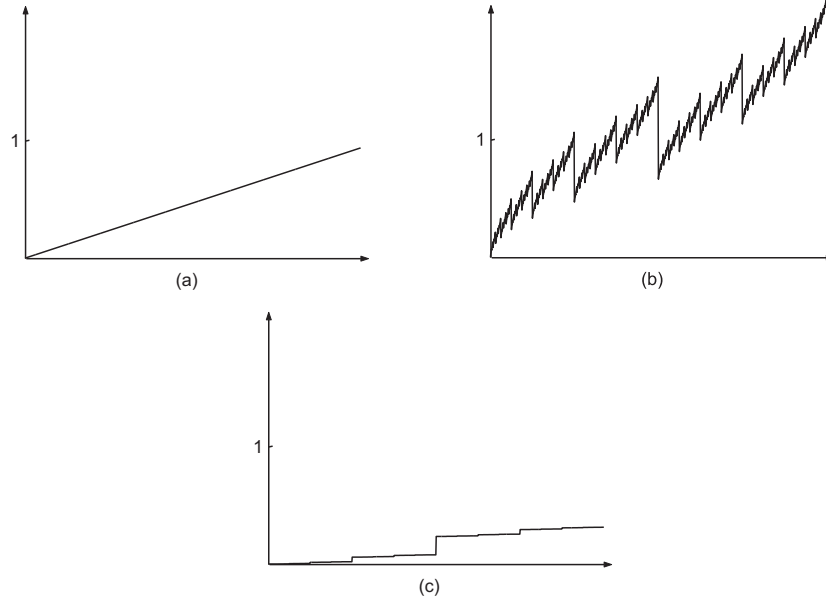


Figure 2.1. The sketch maps of: (a) g on K_2 ; (b) the $1/2$ -order derivative of g on K_2 ; (c) the 1-order integral of g on K_2 .

THEOREM 2.4 $f(x)$ is infinitely integrable; and is an m -order differentiable with $0 \leq m < \frac{\ln 2}{\ln 3}$, and for some character χ of K , for $x \in D$,

$$f^{<m>}(x) = \frac{1}{2} + \sum_{l=1}^{+\infty} \frac{3^{lm}}{6^l} \chi(\beta^{-l}x) \left(\left(\frac{1}{2} + \omega \right) + \left(\frac{1}{2} + \bar{\omega} \right) \chi(\beta^{-l}x) \right) \\ \times \prod_{j=1}^{l-1} (2 - \chi(\beta^{-j}x) - \chi^2(\beta^{-j}x)),$$

for $x \notin D$, $f^{<m>}(x) = 0$. Moreover, the Hausdorff dimension of the image of $f^{<m>}(x)$ with domain D is always 1 for all $-\infty < m \leq \frac{\ln 2}{\ln 3}$.

In [53], we study the Weierstrass type function

$$W(x) = \sum_{k=1}^{+\infty} p^{(s-2)k} \operatorname{Re} \chi(\beta^{-k}x),$$

where $x \in D$, $1 \leq s < 2$, in p -series field.

Then, we have

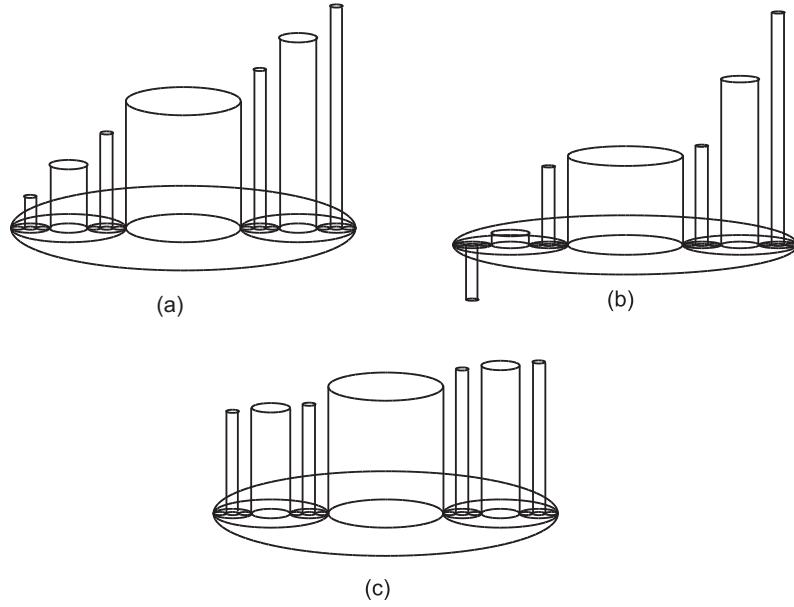


Figure 2.2. The sketch map of: (a) 3-adic Cantor function on K_3 ; (b) 1/2-order derivative of the 3-adic Cantor function; (c) 1-order integral of the 3-adic Cantor function.

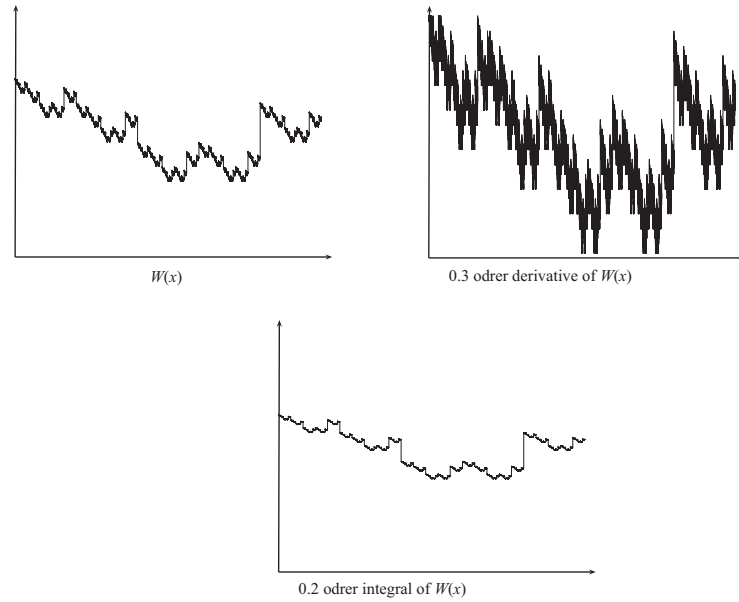


Figure 2.3. The sketch maps of $W(x)$, $W^{<0.3>}(x)$ and $W_{<0.2>}(x)$ for $p = 5$, $s = 1.45$.

THEOREM 2.5 *The function $W(x)$ is infinitely integral and m -order differentiable with $m < 2 - s$, and*

(1) *For the Box dimension $\dim_B$ and Paking dimension $\dim_P$, we have*

$$\dim_B \Gamma(W(x)^{<m>}, D) = \dim_P \Gamma(W(x)^{<m>}, D) = s + m, \quad m \in [1 - s, 2 - s);$$

(2) *For Hausdorff dimension, we have*

$$\dim_H \Gamma(W(x)^{<m>}, D) = s + m$$

where

$$\begin{aligned} m &\in (1 - s, 2 - s), & \text{if } p = 2, \\ m &\in (\log_p(2p - 1) - s, 2 + \log_p y(b_p) - s), & \text{if } p > 2. \end{aligned}$$

This theorem is very interesting and important, since it gives connected relationship between the Gibbs derivatives and fractal dimensions.

More applications of Gibbs derivatives are to medical study, for example, the study of liver's cancer, gene's action in liver's cancer, and so on.

We will continue our study on Gibbs derivatives, in theory and applications.

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